Wiener-Hopf techniques for the analysis of the time-dependent behavior of queues

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Contents

Chapter 1

Introduction

This thesis studies the (numerical) analysis of the time-dependent behavior of some queueing systems based on the Wiener-Hopf factorization technique. The latter technique basically is used to solve Wiener-Hopf integral equations, and is discussed extensively in the books by Corduneanu[19] and Zabreyko[44]. The probabilistic interpretation of these equations is studied by Asmussen[9]. Cohen[18] gives an introduction to the use of Wiener-Hopf equations in queueing theory. This technique is known as a powerful analytic tool for analyzing queueing systems.

To obtain the time-dependent distributions of interest, first we specify the initial workload and then we derive the (systems of) transformed Wiener-Hopf integral equation(s). The (system of) equation(s) is(are) then solved by applying the Wiener-Hopf factorization technique. This approach is motivated by the thesis by Regterschot[38], where the Wiener-Hopf factorization technique is applied to study the steady-state behavior of some queueing systems.

For a queueing system with a non-zero initial workload, the (system of) transformed Wiener-Hopf integral equation(s) contains a term that is related to the initial workload. The solution of the (system of) equation(s) requires a decomposition of the latter term. The need of this decomposition is the main difference between the analysis for the steady-state behavior (in [38]) and the analysis of the time-dependent behavior in the present thesis.

Transform techniques are well known techniques in the analysis of queueing systems. The moments of the time-dependent distribution can be obtained easily by differentiating the transform. The cumulative distribution function and the probability density function can be obtained by inverting the transform. We use the Wiener-Hopf factorization and the decomposition in analyzing the time-dependent behavior of some queueing systems, since this approach will give us explicit expressions for the transforms of the time-dependent distributions of interest which are easy to differentiate in order to obtain the moments. Moreover, the explicit expressions for the transforms enable us to perform numerical inversion in order to obtain the cumulative distribution functions and the probability density functions.

There are many papers in which the transform technique is used to analyze the time-dependent behavior of queueing systems. Papers by Bertsimas et. al.[13, 12] and Tanaka et. al[42] are a few examples. Most authors give the time-dependent behavior in the form of transforms or in the moments of the distributions of interest. The numerical inversion for obtaining cumulative distribution functions or probability density functions are rarely attempted, because it is considered difficult. Fortunately, there are some efforts to develop effective numerical inversion algorithms so that numerical inversion can be easily understood and performed. In particular, the algorithms proposed by Abate & Whitt[3, 2] and Abate, Choudhury, & Whitt[1] are very easy to perform and enable us to do careful error analysis. These effective numerical inversion algorithms and the explicit expressions for the transforms guide us in analyzing the time-dependent distributions of interest, which this thesis is about.

1.1 Focus of the thesis

We focus our analysis on two types of queueing systems: the classical queueing systems and the fluid flow models. In the classical queueing systems the customers are treated individually. We study two classes of queuing systems: the single server $GI/G/1$ system, and the multi server $GI/H_m/s$ system.

To investigate the applicability of our approach to the problems in the areas of computer system modelling and telecommunication system modelling, we study fluid flow models since these models are often used in those areas. The fluid flow model is a queueing system where the input traffic of the system is treated as if it is a fluid, flowing continuously into a buffer, which drains at a constant rate. The input flow is modulated by a (continuous-time) stochastic process, and the input flow rate is constant between transitions of the underlying jump process.

The first fluid flow model studied is the Markovian fluid flow model, where the input flow is modulated by a (continuous-time) Markov chain. The second one is the semi-Markovian fluid flow model, a generalization of the Markovian fluid flow model, where the inter-jump time of the underlying process has a non-exponential distribution.

1.2 Methodology

For the $GI/G/1$ system, we consider the process $\{(W_n, T_n), n = 1, 2, \cdots\}$ defined on the state space $R_+ \times R_+$, in which $\{T_n\}$ is an increasing time sequence generated by the input process and W_n is the actual waiting time of the nth customer who arrives at time T_n.

For $Re(\phi) \geq 0$ and $(|r| \leq 1, Re(\eta) > 0, v \geq 0)$ or $(|r| < 1, Re(\eta) \geq 0, v \geq 0)$, we introduce the generating functions

$$Z(r, \phi, \eta, v) = \sum_{n=1}^{\infty} r^n E\left(e^{-\phi W_n - \eta T_n} | C_0 = v\right), \tag{1.1}$$

where C_0 denotes the initial number of customers. For this system, we have a boundary value problem on the imaginary axis $Re(\phi) = 0$ characterized by the equation

$$Z(r, \phi, \eta, v)(1 - rG(\phi, \eta)) = rZ^0(\phi, \eta, v) + V(r, \phi, \eta, v), \tag{1.2}$$

in which $Z^0(\phi,\eta,v)$ is a function induced by the initial conditions defined for $Re(\phi)\geq 0$, $Re(\eta)\geq 0$, $v\geq 0$, and, $V(z,\phi,\eta,v)$ is a function defined for $Re(\phi)\leq 0$ and ($|r|\leq 1$, $Re(\eta)>0, v\geq 0$) or ($|r|<1, Re(\eta)\geq 0, v\geq 0$).

For the $GI/H_m/s$ system and the fluid flow models, we consider the process $\{(W_n,T_n,X_n),\ n=1,2,\cdots\}$ defined on the state space $R_+\times R_+\times S$, with S a finite set, in which $\{T_n\}$ is an increasing time sequence generated by the input process, W_n can be thought of as workload of the system at time T_n and X_n represents the state at time T_n of a certain stochastic process. In the $GI/H_m/1$ system, this process is the service process. In the fluid flow models, this process is the underlying (semi-)Markov process, which we then denote by the $\{J_t,\geq 0\}$.

Let $\mathbf{1}(A)$ be the indicator function of the event A. Introducing the generating functions

$$Z_{ij}(r,\phi,\eta,v)=\sum_{n=1}^{\infty} r^n E\left(e^{-\phi W_n-\eta T_n}\mathbf{1}(X_n=j)|X_1=i,W_1=v\right),\quad i,j\in S, \tag{1.3}$$

for $Re(\phi)\geq 0$ and ($|r|\leq 1, Re(\eta)>0, v\geq 0$) or ($|r|<1, Re(\eta)\geq 0, v\geq 0$), one is led to solve a boundary value problem on the imaginary axis $Re(\phi)=0$ characterized by a (matrix) equation of the following form

$$\mathbf{Z}(r,\phi,\eta,v)(\mathbf{I}-r\mathbf{G}(\phi,\eta))=r\mathbf{Z}^0(\phi,\eta,v)+\mathbf{V}(r,\phi,\eta,v), \tag{1.4}$$

in which $\mathbf{Z}^0(\phi,\eta,v)$ is a (matrix) function induced by the initial conditions defined for $Re(\phi)\geq 0, Re(\eta)\geq 0$ and, $\mathbf{V}(z,\phi,\eta,v)$ is a (matrix) function defined for $Re(\phi)\leq 0$ and ($|r|\leq 1, Re(\eta)>0, v\geq 0$) or ($|r|<1, Re(\eta)\geq 0, v\geq 0$).

The equations (1.2) and (1.4) are (a system of) Wiener-Hopf integral equations. In the kernel $\mathbf{H}(r,\phi,\eta)=\mathbf{I}-r\mathbf{G}(\phi,\eta)$, $\mathbf{I}$ is the identity matrix and $\mathbf{G}$ is such that $\mathbf{G}(0,0)$ is a stochastic (transition) matrix.

Two steps are necessary to solve (1.2) and (1.4) for $\mathbf{Z}$ as a function of ϕ. In the first step a Wiener-Hopf factorization of the kernel $\mathbf{H}$ has to be found such that

$$\mathbf{H}(r,\phi,\eta)=\mathbf{H}^+(r,\phi,\eta)\mathbf{H}^-(r,\phi,\eta) \tag{1.5}$$

where $\mathbf{H}^+(r,\phi,\eta)$ is non-singular for $Re(\phi)>0$ and satisfies the properties A^+, that is, it is analytic for $Re(\phi)>0$ and continuous and bounded for $Re(\phi)\geq 0$ and $\mathbf{H}^-(r,\phi,\eta)$, is non-singular for $Re(\phi)\leq 0$ and satisfies the properties A^- that is analytic for $Re(\phi)<0$ and continuous and bounded for $Re(\phi)\leq 0$. Having found such a factorization, (1.2) or (1.4) and (1.5) gives on $Re(\phi)=0$ the boundary equation

$$\begin{aligned}&\mathbf{Z}(r,\phi,\eta,v)\mathbf{H}^+(r,\phi,\eta)\\ =&r\mathbf{Z}^0(\phi,\eta,v)\mathbf{H}^-(r,\phi,\eta)^{-1}+\mathbf{V}(r,\phi,\eta,v)\mathbf{H}^-(r,\phi,\eta)^{-1}.\end{aligned} \tag{1.6}$$

The left-hand side satisfies the properties A^+ while the second term on the right satisfies the properties A^-. Now the first term on the right involving $\mathbf{Z}^0(\phi,\eta,v)$ neither satisfies A^+ nor A^-. In the second step we, therefore, have to find a decomposition such that

$$r\mathbf{Z}^0(\phi,\eta,v)\mathbf{H}^-(r,\phi,\eta)^{-1}=\mathbf{K}^+(r,\phi,\eta,v)+\mathbf{K}^-(r,\phi,\eta,v), \tag{1.7}$$

in which $\mathbf{K}^+(r,\phi,\eta,v)$ satisfies properties A^+ and $\mathbf{K}^-(r,\phi,\eta,v)$ satisfies properties A^-. From (1.2) or (1.4) and (1.7) we then obtain on $Re(\phi) = 0$

$$\begin{aligned}&\mathbf{Z}(r,\phi,\eta,v)\mathbf{H}^+(r,\phi,\eta) - \mathbf{K}^+(r,\phi,\eta,v)\\ =&\mathbf{K}^-(r,\phi,\eta,v) + \mathbf{V}(r,\phi,\eta,v)\mathbf{H}^-(r,\phi,\eta)^{-1},\end{aligned} \tag{1.8}$$

in which the function on the left satisfies the properties A^+ and the function on the right satisfies properties A^-. Hence, by analytic continuation we can define a bounded entire function on the whole ϕ-plane, which by Liouville's theorem must be a matrix independent of ϕ, say $\mathbf{C}(r,\eta,v)$. This finally gives, apart from the determination of $\mathbf{C}(r,\eta,v)$, the (formal) solution

$$\mathbf{Z}(r,\phi,\eta,v) = \big(\mathbf{C}(r,\eta,v) + \mathbf{K}^+(r,\phi,\eta,v)\big)\,\mathbf{H}^+(r,\phi,\eta)^{-1}, \tag{1.9}$$

for $Re(\phi) \geq 0$ and $(|r| \leq 1, Re(\eta) > 0, v \geq 0)$ or $(|r| < 1, Re(\eta) \geq 0, v \geq 0)$. With this solution we are able to determine the time-dependent distributions of interest.

It should be noted that, for the models we study in this thesis, the solution $\mathbf{Z}(r,\phi,\eta,v)$ is a rational (matrix) function in ϕ. Then, it is possible to invert $\mathbf{Z}(r,\phi,\eta,v)$ with respect to the variable ϕ analytically.

1.3 Finding the steady-state distributions

Since information on numerical steady state results is desirable when studying numerical solutions to time-dependent equations, we derive the steady state results for all models in this thesis, although most of these results are already known or have been derived in [38] or in de Smit[24, 21, 23].

For the $GI/G/1$ system, if the process $\{W_n\}$ converges (weakly) to a random variable W, then the steady-state distribution of W can be found by applying Abel's limit theorem for generating functions to (1.9). More precisely, the expression for the transform $Z(\phi) = E\left[e^{-\phi W}|C_0 = v\right]$ can be obtained by evaluating $\lim_{z\uparrow 1}(1-z)\mathbf{Z}(z,\phi,0,v)$ that is

$$Z(\phi) = E\left[e^{-\phi W}|C_0 = v\right] = \lim_{z\uparrow 1}(1-z)\mathbf{Z}(z,\phi,0,v).$$

Since the function $Z(\phi)$ is a rational function, we then can invert this transform analytically to obtain the distribution function of W.

Similarly, for the other queueing systems we study in this thesis, if the process $\{W_n, X_n\}$ converges (weakly) to a random vector (W, X), then the steady-state distribution of (W, X) can be found similarly from (1.9). Then, for $i, j = 1, 2, \ldots, N$, $Re(\phi) \geq 0$,

$$\begin{aligned}Z_{ij}(\phi) &= E\left(e^{-\phi W}\mathbf{1}(X = j)|X_1 = i, W_1 = v\right)\\ &= \lim_{z\uparrow 1}(1-z)Z_{ij}(z,\phi,0,v).\end{aligned}$$

It is shown that if $\{W_n, X_n\}$ converges (weakly) to a random vector (W, X), the function $Z_{ij}(\phi)$ is independent of i and we later use the notation $Z_j(\phi)$ instead of $Z_{ij}(\phi)$. The explicit expression for the distribution function

$$F_j(x) = P\{W \leq x,\ X = j\}, \quad j = 1, 2, \ldots, N,$$

then can be obtained by inverting $Z_j(\phi)$ analytically.

Let V_t be the workload of the system at time $t \geq 0$, and let N_t be the number of transitions of the underlying (semi-)Markov process up to time t. For the fluid flow models, if at time t the underlying (semi-)Markov process is in state j, the input flow rate is assumed to be r_j so that the workload V_t satisfies the relation

$$V_t = [W_{N_t} - r_j(t - T_{N_t})]^+, \tag{1.10}$$

where $x^+ = \max\{0, x\}$. The transform

$$Z^*_{ij}(\phi, \eta, v) = \int_0^\infty e^{-\eta t} E\left[e^{-\phi V_t} \mathbf{1}(J_t = j) | X_1 = i, V_1 = v\right] \, dt \tag{1.11}$$

can be obtained in terms of $Z_{ij}(\phi, \eta, v)$ through a simple analysis of the process $\{(V_t, J_t),\ t \geq 0\}$ and a contour integration. If the weak limit of $\{(V_t, J_t),\ t \geq 0\}$ exists and is denoted by (V, J), then the transform

$$Z^*_j(\phi) = E\left[e^{-\phi V} \mathbf{1}(J = j)\right] \tag{1.12}$$

can be obtained by applying Abel's limit theorem to $Z^*_{ij}(\phi, \eta, v)$, and inverting it analytically will yield the distribution function

$$F^*_j(x) = P\{V \leq j,\ \mathbf{1}(J = j)\}.$$

Moreover, for the classical queueing system $GI/G/1$ the workload V_t satisfies the relation

$$V_t = [W_{N_t} + V_{N_t} - (t - T_{N_t})]^+. \tag{1.13}$$

The relation (1.13), in a similar way as for the fluid flow models, leads to an expression for the transform

$$Z^*(\phi, \eta, v) = \int_0^\infty e^{-\eta t} E[e^{-\phi V_t} | C_0 = v] \, dt \tag{1.14}$$

in terms of $\mathbf{Z}(1, \phi, \eta, v)$. Similarly, we also can derive expressions for the transforms

$$U(r, s, v) = \sum_{n=0}^{\infty} r^n E[s^{C_n} | C_0 = v]$$

and $\int_0^\infty e^{-\eta t} E[s^{C^*_t}] \, dt$, where C_n and C^*_t denote the number of customers in the system just before the arrival of the nth customer and the number of customers in the system at time t, respectively. By applying Abel's limit theorem to these transforms, we obtain the distributions of the number of customers at arrival epochs as well as in continuous times. For the $GI/H_m/s$ system, the distributions of the queue length at arrival epochs and in continuous time are studied in a similar way.

1.4 Finding the time-dependent distributions

It should be noted that the decomposition step in the procedure described in section 1.2 is essentially due to the presence of the $r\mathbf{Z}^0(\phi, \eta, v)$ term in (1.2) and (1.4) and is characteristic for finding the transforms of time-dependent probability distributions. This can be seen from (1.2) and (1.4), with $\eta = 0$, when one applies Abel's limit theorem to equation (1.2) and (1.4), under the provision that $\{W_n\}$ or $\{W_n, X_n\}$ converges weakly as $n \to \infty$, since multiplying (1.2) and (1.4) by $(1 - r)$ the term involving $r\mathbf{Z}^0(\phi, \eta, v)$ will tend to zero as one takes the limit for $r \uparrow 1$.

The time-dependent distributions of interest can be obtained by inverting their multidimensional transforms numerically as proposed by Abate, Choudhury and Whitt[1], Choudhury, Lucantoni and Whitt[16], and Moorthy[35, 36]. The numerical inversion algorithm in [1] is based on the connection between the Laguere-series representation of the function one wants to obtain and its multidimensional Laplace transform. To accelerate the convergence, the algorithm is complemented by the scaling technique, which for inverting the one dimensional transform is effective (see Choudhury, Lucantoni and Whitt[16]).

The transforms we derive in this thesis are not all multidimensional Laplace transform. For example, to obtain the time-dependent distribution of the workload at arrival epochs, W_n, we have to invert the transform $Z(r, \phi, 0, v)$ which is the generating function of the Laplace transform of W_n. The numerical inversion algorithms in [1, 16, 35, 36] can not be applied directly to the transform $Z(r, \phi, 0, v)$. In obtaining time distributions of interest, in this thesis we use a different approach. Noting that the transform $\mathbf{Z}(r, \phi, \eta, v)$ is a rational (matrix) function in ϕ, first we apply an analytic inversion to the transforms. The result is not a rational function anymore, so then we apply numerical inversion.

In the fluid flow models, we assume that the inter-jump times of the underlying process have a common distribution where its Laplace-Stieltjes transform is a rational function. It follows that the kernel $\mathbf{H}(r, \phi, \eta)$ is a rational function in the variable ϕ. The location of the zeros and the poles of $\mathbf{H}(r, \phi, \eta)$ in the complex plane ϕ will guide us in finding the factors $\mathbf{H}^+(r, \phi, \eta)$ and $\mathbf{H}^-(r, \phi, \eta)$, and the Wiener-Hopf factorization used will give us rational factors in ϕ. Furthermore, the expression of $\mathbf{Z}(r, \phi, \eta)$ in (1.9) with respect to the variable ϕ consists of some rational functions and multiplication of rational functions and exponential functions. This enable us to invert $\mathbf{Z}(r, \phi, \eta)$ analytically with respect to the variable ϕ. Let $\mathbf{z}(r, x, \eta)$ be the result of this inversion.

The time-dependent distribution of the workload at transition epochs can be obtained by inverting the generating function $\mathbf{z}(r, 0, x)$. We then apply the numerical inversion algorithm proposed in Abate and Whitt[3] to invert $\mathbf{z}(r, \eta, x)$ since $\mathbf{z}(r, \eta, x)$ is not a simple function to be inverted analytically.

The time-dependent distribution of the workload can be obtained in a similar way. Notice that in obtaining this distribution, the analytical inversion will yield a Laplace-Stieltjes transform which is also not simple to be inverted analytically. The numerical inversion algorithm for inverting the Laplace transform in [3] can be applied to obtain the distribution.

For the classical queueing systems we study in this thesis, the rationality of the kernel $\mathbf{H}(r, \phi, \eta)$ is ensured if the inter-arrival times or the service times have a rational

Laplace-Stieltjes transform. The $GI/H_m/s$ system has this characteristic so that the time-dependent distributions of interest can be obtained by applying the same technique as for the fluid flow models. For the $GI/G/1$ system, we restrict our analysis to the special cases $GI/K_m/1$ and $K_m/G/1$. This restriction is not too strong since the set of distributions with rational Laplace-Stieltjes transform is dense in the distribution space so that any single server queueing system can be approximated by the $GI/K_m/1$ system or the $K_m/G/1$ system.

As mentioned above the numerical inversion is the last crucial step in obtaining the time-dependent distributions. The numerical inversion algorithms proposed in [1] are very useful for inverting the time-dependent distribution functions (more explanation on the algorithms can be found in section 2.4).

1.5 Organization of the thesis

This thesis is organized as follows. After the introduction, in chapter 2 we recall some results from complex function theory and some isolated lemmas and introduce notation that will be used in the sequel. Moreover, we give a brief introduction on the Wiener-Hopf factorization and its application. This chapter also presents the numerical inversion algorithms in [3] and an explanation of how we set the accuracy. In chapter 3 we apply the Wiener-Hopf factorization technique to study the time-dependent behavior of the system $GI/G/1$. In chapter 4 we apply the technique to study the system $GI/H_m/s$. In both chapters, we successfully obtain the time-dependent distributions of the actual waiting times, the virtual waiting times, the number of customers at arrival epochs as well as in continuous time. In chapter 4 we also obtain the time dependent distributions of the queue length at arrival epochs as well as in continuous time. In chapter 5 we study the time-dependent buffer content in the Markovian Fluid Flow Model. The generalization of this model to the semi-Markovian case is studied in chapter 6.

Chapter 2

Some Mathematical Preliminaries

In this chapter, we give some preliminaries that are needed for the analysis in chapter 3 until chapter 6. We begin with section 2.1, which gives us definitions on some contours and identities, and followed by a short discussion of the analytic continuation in section 2.2. In section 2.3 we give a short introduction to Wiener-Hopf factorization, which plays a key role in solving the main systems of equations we derive in chapters 3 - 6. We end this chapter by giving the numerical algorithms used for inverting Laplace-Stieltjes transforms and generating functions.

2.1 Contours and Identities

In this thesis, we will often consider the following contours.

Definition 2.1.1
For $R > \delta \geq 0$, $C^{+}_{\delta,R}$ *is the closed contour consisting of*

1. *the part of the line* $Re(\phi) = -\delta$, *running from* $-\delta + i\sqrt{R^2 - \delta^2}$ *to* $-\delta - i\sqrt{R^2 - \delta^2}$ *and*

2. *the part of circle* $|\phi| = R$, *running counterclockwise from* $-\delta - i\sqrt{R^2 - \delta^2}$ *to* $-\delta + i\sqrt{R^2 - \delta^2}$.

$C^{-}_{\delta,R}$ *is the closed contour consisting of*

1. *the part of the line* $Re(\phi) = -\delta$, *running from* $-\delta - i\sqrt{R^2 - \delta^2}$ *to* $-\delta + i\sqrt{R^2 - \delta^2}$ *and*

2. *the part of circle* $|\phi| = R$, *running counterclockwise from* $-\delta + i\sqrt{R^2 - \delta^2}$ *to* $-\delta - i\sqrt{R^2 - \delta^2}$.

The definition is illustrated by figure 2.1.

For derivations, we use some identities of which the proof can be found in the book by Cohen [17]. First, we introduce the notations

$$[x]^+ = \max(0, x), \quad [x]^- = \min(0, x), \quad -\infty < x < \infty. \tag{2.1}$$

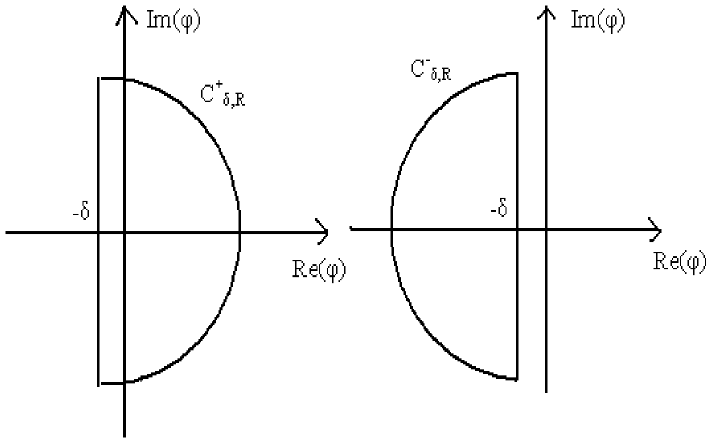

Figure 2.1: The contour $C^{+}_{\delta,R}$ (left) and the contour $C^{-}_{\delta,R}$(right).

Identity 2.1.1
For arbitrary real x and complex numbers ϕ_1 and ϕ_2,

$$e^{-\phi_1[x]^+} + e^{-\phi_2[x]^-} = e^{-\phi_1[x]^+ - \phi_2[x]^-} + 1. \tag{2.2}$$

Proof. See page 142 of Cohen [17].

Identity 2.1.2
For arbitrary real x,

$$e^{-\phi[x]^+} = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{-iR+0}^{iR+0} d\xi \frac{\phi}{\xi(\phi-\xi)} e^{-\xi x}. \tag{2.3}$$

Proof. See page 269 of Cohen [17].

The following identity is the Dirichlet integral representation of the normalized unit step function.

Identity 2.1.3

$$\mathbf{1}(x<0) + \frac{1}{2}\mathbf{1}(x=0) = \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} e^{-\xi x}. \tag{2.4}$$

Proof. See Widder[43].

2.2 Analytic Function and Analytic Continuation

In chapters 3-6 we will consider some analytic functions that satisfy a certain property, which is formulated in the following.

Definition 2.2.1
We say a function f satisfies property A^+ if $f(\phi)$ is
(i) analytic on $Re(\phi) > 0$,
(ii) continuous and bounded on $Re(\phi) \geq 0$,
and we say it satisfies property $\tilde{A}^+$ if, in addition, it is
(iii) bounded away from 0 on $Re(\phi) \geq 0$.
We say a function f satisfies property A^- if $f(\phi)$ is
(i) analytic on $Re(\phi) < 0$,
(ii) continuous and bounded on $Re(\phi) \leq 0$,
and we say it satisfies property $\tilde{A}^-$ if, in addition, it is
(iii) bounded away from 0 on $Re(\phi) \leq 0$.

Next, we recall a theorem called the *principle of analytic continuation.* We will use this theorem in proving some main theorems in this thesis.

Theorem 2.2.1
Let an analytic function $f_1(z)$ be defined in a region Ω_1 and let Ω_2 be another region which has a certain subregion ω, but only this one, in common with Ω_1. Then, if a function $f_2(z)$ exists which is analytic in Ω_2 and coincides with $f_1(z)$ in ω, there can only be one such function. $f_1(z)$ and $f_2(z)$ are called analytic continuations of each other.

Proof. See Knopp [30].

2.3 Wiener-Hopf factorization

The technique to solve the problems in this thesis is based on Wiener-Hopf factorization. In this section we recall some definitions and some theorems about this factorization and its application to the problems in chapters 3 until chapter 6.

Let f, g and k_0 be functions of bounded variation on the real line $(-\infty, \infty)$, where f and h have non-negative support, i.e. $f(t) = k_0(t) = 0$ for $t < 0$. The function f defines a Stieltjes measure df(.) on the positive half-axis $[0, \infty)$ which is used to define Riemann-Stieltjes integrals. The integral equation for f,

$$f(t) - \int_{0-}^{+\infty} g(t-y)df(y) = k_0(t), t \geq 0, \tag{2.5}$$

is called the *Wiener-Hopf integral equation.* Since $f(t) = 0$ for $t < 0$ we may extend this equation to the negative half-axis by introducing a function

$$k(t) = \begin{cases} -\int_{0^-}^{+\infty} g(t-y)df(y), & t < 0, \\ \\ k_0(t), & t \geq 0. \end{cases} \tag{2.6}$$

So we get the *extended Wiener-Hopf integral equation*

$$f(t) - \int_{0^-}^{+\infty} g(t-y)df(y) = k(t),\ -\infty < t < \infty. \tag{2.7}$$

By introducing the Laplace-Stieltjes transforms

$$F(\phi) = \int_{0-}^{+\infty} e^{-\phi t} df(t), \quad Re(\phi) \geq 0,$$
$$G(\phi) = \int_{-\infty}^{+\infty} e^{-\phi t} dg(t), \quad Re(\phi) = 0,$$
$$K(\phi) = \int_{-\infty}^{0^-} e^{-\phi t} dk(t), \quad Re(\phi) \leq 0, \quad \text{and}$$
$$K^0(\phi) = \int_{0^-}^{+\infty} e^{-\phi t} dk_0(t), \quad Re(\phi) \geq 0,$$

and applying transforms to equation (2.7) gives the *transformed Wiener-Hopf equation*

$$F(\phi)(1 - G(\phi)) = K^0(\phi) + K(\phi), \quad Re(\phi) = 0, \tag{2.8}$$

where $F(\phi)$ and $K^0(\phi)$ satisfy the property A^+ and $K(\phi)$ satisfies the property A^-. The equation (2.8) can be solved using a factorization method applied to the symbol of (2.8)

$$H(\phi) = 1 - G(\phi).$$

The factorization is referred to as Wiener-Hopf factorization since it is connected to the Wiener-Hopf technique in the theory of integral equations. This technique is about to write a complex valued function $H(\phi)$, which is bounded and continuous on $Re(\phi) = 0$ with $\lim_{\phi \to i\infty} H(\phi) = \lim_{\phi \to -i\infty} H(\phi) = 1$, in the form

$$H(\phi) = H^+(\phi)H^-(\phi), \quad Re(\phi) = 0, \tag{2.9}$$

where $H^+(\phi)$ satisfies property A^+ and $H^-(\phi)$ satisfies property A^-.

We shall only consider factorizations with

$$H^+(+i\infty) = H^+(-i\infty) = H^-(+i\infty) = H^-(-i\infty) = 1.$$

Since both factors are bounded at infinity and analytic in their respective half-planes $Re(\phi) > 0$ and $Re(\phi) < 0$ they are bounded in the closed half-planes $Re(\phi) \geq 0$ and $Re(\phi) \leq 0$ respectively. We impose the condition that $H(\phi)$ does not vanish on the imaginary axis, i.e.

$$H(\phi) \neq 0, \quad Re(\phi) = 0. \tag{2.10}$$

This condition implies that $H^+(\phi)$ and $H^-(\phi)$ can not vanish on the imaginary axis $Re(\phi) = 0$. The factorization (2.9) is *regular* if at least one of the factors $H^+(\phi)$ and $H^-(\phi)$ does not vanish in the half-plane of analyticity. It is *canonical* if both factors $H^+(\phi)$ and $H^-(\phi)$ do not vanish in their half-planes of analyticity, so that

$$\begin{aligned} H^+(\phi) &\neq 0, \quad Re(\phi) \geq 0, \text{ and} \\ H^-(\phi) &\neq 0, \quad Re(\phi) \leq 0. \end{aligned} \tag{2.11}$$

The existence of the canonical factorization (2.11) is given by the following theorem.

Theorem 2.3.1 (Scalar factorization theorem)
A function $H(\phi) = 1 - G(\phi)$ admits a canonical factorization if and only if

- $H(\phi) \neq 0, \quad Re(\phi) = 0,$
- *one of the following is satisfied*
  1. *the number of zeros of $H(\phi)$ in $Re(\phi) > 0$ is equal to the number of poles of $H(\phi)$ in $Re(\phi) > 0$,*
  2. *the number of zeros of $H(\phi)$ in $Re(\phi) < 0$ is equal to the number of poles of $H(\phi)$ in $Re(\phi) < 0$.*

The canonical factorization is unique. Moreover we have

$$H^+(\phi) = 1 + \int_0^{+\infty} e^{-\phi t} dC(t), \quad Re(\phi) \geq 0, \text{ and}$$
$$H^-(\phi) = 1 + \int_{-\infty}^{0} e^{-\phi t} dC(t), \quad Re(\phi) \leq 0,$$

where $C(t)$ is a function of bounded variation on the real line.

Proof. See Corduneanu [19] and Regterschot [38]. ■

The conditions 1 and 2 in Theorem 2.3.1 can be verified by Rouché's theorem. If in a half-plane the number of zeros is not equal to the number of poles then there exists a factorization as well. However it is not canonical and not necessarily unique. For more details see Corduneanu [19] and Zabreyko [44].

In chapters 4-6, we are dealing with some systems of transformed Wiener-Hopf equations. The necessary and sufficient conditions for these systems to admit a canonical factorization is ensured in Bart, Gohberg, and Kaashoek[10]. The conditions involve the $\det H(\phi)$ instead of $H(\phi)$, and we can use a generalization of Rouché's theorem, given in de Smit [21], to verify the conditions.

If the canonical factors for the symbol of (2.8) exist then we have

$$F(\phi)H^+(\phi) = K^0(\phi)/H^-(\phi) + K(\phi)/H^-(\phi), \quad Re(\phi) = 0. \tag{2.12}$$

Now the left hand-side satisfies property A^+ and the last term of the right hand-side satisfies property A^-. We then try to find a decomposition of $K^0(\phi)/H^-(\phi)$, i.e. we look for two functions $C^+(\phi)$ and $C^-(\phi)$ such that

- $C^+(\phi)$ satisfies property A^+,
- $C^-(\phi)$ satisfies property A^-,
- $K^0(\phi)/H^-(\phi) = C^+(\phi) + C^-(\phi)$.

With this decomposition we have from (2.12)

$$F(\phi)H^+(\phi) - C^+(\phi) = C^-(\phi) + K(\phi)/H^-(\phi). \tag{2.13}$$

At this point we invoke Liouville's theorem.

Theorem 2.3.2 (Liouville's theorem)
A function analytic and bounded in the whole complex plane is constant.

Proof. See page 451 of Apostol[5]. ■

From (2.13) it now follows by analytic continuation that it is possible to define a function equal to the left hand-side of (2.13) for $Re(\phi) \geq 0$ and equal to the right hand-side of (2.13) for $Re(\phi) \leq 0$. It now follows from Liouville's theorem that

$$F(\phi)H^+(\phi) - C^+(\phi) = \text{constant}, \quad Re(\phi) \geq 0, \tag{2.14}$$

and

$$C^-(\phi) + K(\phi)/H^-(\phi) = \text{constant}, \quad Re(\phi) \leq 0, \tag{2.15}$$

where the constant is determined by the known value, c say, at the origin, so

$$c = F(0)H^+(0) - C^+(0) = C^-(0) + K(0)/H^-(0). \tag{2.16}$$

Since the factorization is unique we now have

Theorem 2.3.3
If the function $H(\phi) = 1 - G(\phi)$ admits a canonical factorization, then equation (2.8) has the unique solution

$$F(\phi) = \left(C^+(\phi) + F(0)H^+(0) - C^+(0)\right)/H^+(\phi), \quad Re(\phi) \geq 0, \tag{2.17}$$

and

$$K(\phi) = \left(F(0)H^+(0) - K^+(0) - C^-(\phi)\right)H^-(\phi), \quad Re(\phi) \leq 0. \tag{2.18}$$

Proof. Equations (2.17) and (2.18) are obtained directly by substituting (2.16) into (2.14) and (2.15). ■

2.4 Numerical inversions

In this section we discuss the numerical inversions for the Laplace transforms and probability generating functions proposed in Abate & Whitt[3].

2.4.1 Numerical inversion algorithm for Laplace transforms

Given the Laplace transform

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t)dt, \qquad Re(s) \geq 0, \tag{2.19}$$

where f is a function on the positive real line, we want to invert (2.19) to obtain the function f. The analytical formula for this function is given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itu} \hat{f}(u)du, \quad t \in \mathbb{R}^+, \tag{2.20}$$

but often the integral in (2.20) is difficult to evaluate analytically. A numerical inversion is then appropriate. The numerical inversion is based on the integral (2.20), in which the integral is evaluated numerically by using the trapezoidal rule. It yields an approximation for $f(t)$ in terms of the alternating series

$$f(t) \approx \frac{e^{A/2}}{2t} Re(\hat{f}) \left(\frac{A}{2t} \right) + \frac{e^{A/2}}{t} \sum_{k=1}^{\infty} (-1)^k Re(\hat{f}) \left(\frac{A + 2k\pi i}{2t} \right), \tag{2.21}$$

where real number A and integer numbers m and n are parameters to control the accuracy. The series is then approximated by the *Euler sum*

$$E(t, m, n) = \sum_{k=0}^{m} \binom{m}{k} 2^{-m} S_{n+k}(t) \tag{2.22}$$

where

$$S_n(t) = \sum_{k=0}^{n} (-1)^k a_k(t), \tag{2.23}$$

with

$$a_0(t) = \hat{f} \left(\frac{A}{2t} \right) /2, \tag{2.24}$$

$$a_k(t) = Re(\hat{f}) \left(\frac{A + 2k\pi i}{2t} \right), \quad k \geq 1, \tag{2.25}$$

so that

$$f(t) \approx \frac{e^{A/2}}{t} E(t, m, n). \tag{2.26}$$

In [3] it is shown that $|E(t, m, n) - E(t, m, n+1)|$ can be used for estimating the error due to the approximation formula (2.26). It is indicated that to obtain accuracy to 10^{-7}, we can set $A = 19.1$, $m = 11$, and $n = 15$.

2.4.2 Numerical inversion algorithm for generating functions

Suppose that

$$g(z) = \sum_{j=0}^{\infty} z^j P(X=j) = \sum_{j=0}^{\infty} z^j p_j, \quad |z| \le 1, \tag{2.27}$$

the probability generating function of a random variable with non-negative integer values X, is given. The analytical formula for $p_j = P(X=j)$ is

$$p_j = \frac{1}{2\pi i}\int_{C_r} \frac{g(z)}{z^{j+1}} dz, \tag{2.28}$$

where C_r is the circle with center at origin and of radius $r, 0 < r < 1$, and the integration is taken counter clockwise. Let $z = re^{iu}$. Substituting this to (2.28) we obtain

$$\begin{aligned} p_j &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{g(re^{iu})}{(re^{iu})^{j+1}} ire^{iu} du \\ &= \frac{1}{2\pi r^j}\int_0^{2\pi} g(re^{iu})e^{-iju} du \\ &= \frac{1}{2\pi r^j}\int_0^{2\pi} \left(Re(g(re^{iu})) + iIm(g(re^{iu}))\right)(\cos(ju) - i\sin(ju))\, du. \end{aligned} \tag{2.29}$$

Since p_j is a real number, then

$$p_j = \frac{1}{2\pi r^j}\int_0^{2\pi} \left(\cos(ju)Re(g(re^{iu})) + \sin(ju)Im(g(re^{iu}))\right) du. \tag{2.30}$$

The trapezoidal rule with step size π/j is then applied to approximate the integral in (2.30), and yields

$$\begin{aligned} p_j &\approx \frac{\pi}{2\pi j r^j}\left[\frac{g(r)+g(-r)}{2} + \sum_{k=1}^{2j-1}\cos(k\pi)Re(g(re^{ik\pi/j})) + \sum_{k=1}^{2j}\sin(k\pi)Im(g(re^{ik\pi/j}))\right] \\ &= \frac{1}{2jr^j}\sum_{k=1}^{2j}(-1)^k Re(g(re^{ik\pi/j})), \end{aligned}$$

and with some algebra we obtain

$$p_j \approx \frac{1}{2jr^j}\left[g(r) + g(-r) + 2\sum_{k=1}^{j-1}(-1)^k Re(g(re^{ik\pi/j}))\right]. \tag{2.31}$$

Denote the right hand side of (2.31) by $\tilde{p}_j$. In [3] it is proven that for $0 < r < 1$ and $j \ge 1$,

$$|p_j - \tilde{p}_j| \le \frac{r^{2j}}{1-r^{2j}}.$$

But for practical purposes, we can think of the error bound as r^{2j} since $\frac{r^{2j}}{1-r^{2j}}$ is approximately equal to r^{2j} when r^{2j} is small. Hence, to have accuracy to $10^{-\gamma}$, we let $r = 10^{-\gamma/2j}$.

Chapter 3

The Single Server $GI/G/1$ queue

3.1 Introduction

We consider a single server queueing system with renewal input and infinite waiting room in which customers are served in order of arrival, i.e. with first come - first served (FCFS) discipline. We choose $t = T_0 = 0$ at the arrival epoch of an arbitrary customer. We assume that this customer finds upon his arrival C_0 other customers in the system, which are numbered $1, 2, \ldots, C_0$ in order of their arrival. These customers will be referred to as special customers. For convenience we assume that the first special customer enters service at $t = 0$. The service times of the special customers will be denoted by $X_1, X_2, \ldots, X_{C_0}$. After the arrival at time T_0 subsequent customers arrive at time epochs $T_1, T_2, \ldots$. The inter-arrival times are denoted by $A_n = T_n - T_{n-1}, n = 1, 2, \ldots$ and the service time of the nth customer is denoted by $B_n, n = 0, 1, \ldots$. By assumption $\{A_n\}$ constitutes a sequence of independent identically distributed (i.i.d.) nonnegative random variables with

$$F(x) \quad = P(A_n \leq x)$$

$$F(0+) = 0$$

$$E(A_n) = \alpha < \infty.$$

Also $\{B_n\}$ are i.i.d. nonnegative random variables and we denote

$$G(x) \quad = P(B_n \leq x)$$

$$\text{with} \quad G(0+) = 0$$

$$E(B_n) = \beta < \infty.$$

We assume that the probability distribution of $A_n, n = 1, 2, \cdots$, is non-lattice. Moverover, we assume that $\{A_n\}, \{B_n\}$ and $\{X_i, i = 1, 2, \ldots, C_0\}$ are three independent families of random variables. As usual, the traffic intensity ρ is defined by β/α.

We are interested in the steady state (if it exists) and time dependent probability distributions of the actual waiting time of the nth customer, the virtual waiting time at time t and, the number of customers in the system at arrival epochs and in continuous time.

These system characteristics have been investigated by Cohen[17], Bertsimas *et al.* [13] and Bertsimas & Nakazato[12], under the assumption $C_0 = 0$.

In [13] the analysis is done by solving a Hilbert factorization problem. Two special cases of the problem, i.e. the cases in which either the probability distribution of the inter-arrival times or the service times has a rational Laplace transform are solved explicitly, yielding simple closed-form expressions for the Laplace transforms of the waiting time distribution and the busy period distribution. Algorithmically, the approach offers a method for finding these distributions through numerical inversion, which is claimed to be very tractable.

The two special cases mentioned above are also studied in de Smit[24] for the steady state. Wiener-Hopf factorization is used to analyze the problem, and as a result, the Laplace-Stieltjes transforms of the steady-state distribution of waiting time of nth customer and the distribution of the virtual waiting time are obtained.

Furthermore, in [12] another special case of the model is considered. This special case is the $MGE_L/MGE_M/1$ queue, that is the queueing model in which the inter-arrival times and the service times have a mixed generalized Erlang distribution. The authors use the method of stages, and give closed-form expressions for the Laplace transforms of the queue length distribution and the waiting time distribution. Some examples of the distributions of the busy period, the queue length, and the waiting time are given, obtained through numerical inversion of the Laplace transforms.

For the analysis of the present model, we use the same method as in [24]. To find the distribution function of the waiting time of the nth arbitrary customer, Wiener-Hopf factorization is used. Later we will see that this factorization must be followed by a decomposition of a certain function since in our model we have a non-zero waiting time for the customer who arrives at $t = 0$. For the two special cases studied in [13], which we denote by $GI/K_n/1$ and $K_m/G/1$, an explicit factorization can be found. This gives us an explicit expression for the generating function of the Laplace-Stieltjes transform of the distribution of the actual waiting time of the nth customer. Based on this result, we could derive an explicit expression for the Laplace-Stieltjes transform of the virtual waiting time.

For the study of the number of customers in the system, we derive a general expression for the Laplace-Stieltjes transform of the time-dependent expectation of the number of customers using contour integration. For the systems $GI/K_n/1$ and $K_m/G/1$, the expression for the Laplace-Stieltjes transforms can be determined explicitly. The explicit expressions for the transforms enable us to perform a numerical inversion of these transforms to obtain the time-dependent distributions/expectations of interest. We apply the numerical inversion algorithm proposed in [3], and the numerical results can be found in the end of this chapter.

This chapter is organized as follows. After giving some notations and definitions in section 3.2, we will study the probability distribution of the actual waiting time of the nth customer in section 3.3. Then in section 3.4 we derive the probability distribution of the virtual waiting time. Based upon some results in sections 3.3 and 3.4, we subsequently study the number of customers at arrival epochs in section 3.5 and for continuous time in section 3.6. A more detailed study of these distributions for the queueing models $GI/K_n/1$ and $K_m/G/1$, can be found in section 3.7. In section 3.7.3 we give some examples of the distributions obtained by numerical inversion. For the systems with traffic intensity $\rho < 1$

we give the distributions in steady state as well as in transient state and, for the systems with $\rho > 1$, we give the time dependent distribution of the number of customers at time t and its behavior as t increases.

3.2 Notations and definitions

We denote the actual waiting time of nth customer by W_n and the virtual waiting time at time t by V_t. If $C_0 = \gamma$, then $W_0 = \sum_{i=1}^{\gamma} X_i$. We assume that the X_i have finite positive mean and, that their probability distribution is non-lattice and has a rational Laplace-Stieltjes transform of the following form

$$\mathcal{P}(\phi) = E\left[e^{-\phi X_i}\right] = \frac{P(\phi)}{\prod_{i=1}^{k}(\phi + w_i)}, \tag{3.1}$$

Consequently,

$$E\left[e^{-\phi W_0} | C_0 = \gamma\right] = \frac{P^{\gamma}(\phi)}{\prod_{i=1}^{k}(\phi + w_i)^{\gamma}}, \tag{3.2}$$

where $Re(w_i) > 0, \quad i = 1, 2, \ldots, k$, and in which $P(\phi)$ is a polynomial of degree $k - 1$ or less. We assume that the coefficient of ϕ^d, where d is the degree of $P(\phi)$, is unity.

Let the L-S transforms of the distribution functions of the inter-arrival and service times be denoted by

$$A(\phi) = \int_0^{\infty} e^{-\phi x} F(dx), \quad Re(\phi) \geq 0$$

and

$$B(\phi) = \int_0^{\infty} e^{-\phi x} G(dx), \quad Re(\phi) \geq 0,$$

respectively. We assume that there exists a $\delta > 0$ such that $A(\phi)$ and $B(\phi)$ can be continued analytically into the region $Re(\phi) > -\delta$.

3.3 The distribution of actual waiting times

Since the service discipline is FCFS, the actual waiting times satisfy the recurrence relation

$$W_{n+1} = [W_n + B_n - A_{n+1}]^+ \quad n = 0, 1, \ldots.$$

Let for $(|r| < 1, Re(\phi) \geq 0, Re(\eta) \geq 0, \gamma \geq 0)$, or $(|r| \leq 1, Re(\phi) \geq 0, Re(\eta) > 0, \gamma \geq 0)$, or $(|r| \leq 1, Re(\phi) > 0, Re(\eta) \geq 0, \gamma \geq 0)$,

$$Z(r, \phi, \eta, \gamma) = \sum_{n=0}^{\infty} r^n E\left[e^{-\phi W_n - \eta T_n} | C_0 = \gamma\right],$$

and let

$$V(r, \phi, \eta, \gamma) = \sum_{n=0}^{\infty} r^{n+1} E\left[\left(1 - e^{-\phi [W_n + B_n - A_{n-1}]^-}\right) e^{-\eta T_{n+1}} | C_0 = \gamma\right],$$

for $(|r| < 1, Re(\phi) \leq 0, Re(\eta) \geq 0, \gamma \geq 0)$ or $(|r| \leq 1, Re(\phi) \leq 0, Re(\eta) > 0, \gamma \geq 0)$.

Theorem 3.3.1
For $(|r| < 1, Re(\phi) = 0, Re(\eta) \geq 0, \gamma \geq 0)$ *or* $(|r| \leq 1, Re(\phi) = 0, Re(\eta) > 0, \gamma \geq 0)$,

$$Z(r, \phi, \eta, \gamma)\{1 - rA(\eta - \phi)B(\phi)\} = \frac{P^{\gamma}(\phi)}{\prod_{i=1}^{k}(\phi + w_i)^{\gamma}} + V(r, \phi, \eta, \gamma). \tag{3.3}$$

Proof. By using the identity 2.1.1 with $\phi_1 = \phi_2 = \phi$, that is

$$e^{-\phi x^{+}} = e^{-\phi x} + 1 - e^{-\phi x^{-}} \tag{3.4}$$

we have for $Re(\phi) = 0$, $Re(\eta) \geq 0$, and $\gamma \geq 0$,

$$\begin{aligned} & E\left[e^{-\phi W_{n+1} - \eta T_{n+1}} | C_0 = \gamma\right] \\ = \; & E\left[e^{-\phi[W_n + B_n - A_{n+1}]^{+} - \eta T_{n+1}} | C_0 = \gamma\right] \\ = \; & E\left[e^{-\phi[W_n + B_n - A_{n+1}] - \eta T_{n+1}} | C_0 = \gamma\right] \\ & + E\left[\left(1 - e^{-\phi[W_n + B_n - A_{n+1}]^{-}}\right) e^{-\eta T_{n+1}} | C_0 = \gamma\right] \\ = \; & E\left[e^{-\phi W_n - \eta T_n} | C_0 = \gamma\right] E\left[e^{-\phi B_n - (\eta - \phi) A_{n+1}} | C_0 = \gamma\right] \\ & + E\left[\left(1 - e^{-\phi[W_n + B_n - A_{n+1}]^{-}}\right) e^{-\eta T_{n+1}} | C_0 = \gamma\right], \end{aligned}$$

using the independence assumptions and the fact that $T_{n+1} = T_n + A_{n+1}$. If we multiply by r^{n+1} and sum over n this yields for $Re(\phi) = 0$ and $(|r| < 1, Re(\eta) \geq 0, \gamma \geq 0)$ or $(|r| \leq 1, Re(\eta) > 0, \gamma \geq 0)$,

$$Z(r, \phi, \eta, \gamma) - E\left[e^{-\phi W_0} | C_0 = \gamma\right] = rZ(r, \phi, \eta, \gamma)A(\eta - \phi)B(\phi) + V(r, \phi, \eta, \gamma)$$

noting that $T_0 = 0$ and using the independence of the service times and inter-arrival times and, we get (3.3), using (3.2).

■

It can be shown, see Cohen[17], that for fixed $(|r| < 1, Re(\eta) \geq 0)$ or $(|r| \leq 1, Re(\eta) > 0)$, the function $1 - rA(\eta - \phi)B(\phi)$ can be factorized, i.e. for $Re(\phi) = 0$,

$$1 - rA(\eta - \phi)B(\phi) = K^{+}(r, \phi, \eta)K^{-}(r, \phi, \eta), \tag{3.5}$$

where, in the complex ϕ plane, $K^{+}(r, \phi, \eta)$ satisfies conditions $\tilde{A}^{+}$ and $K^{-}(r, \phi, \eta)$ satisfies conditions $\tilde{A}^{-}$. Then, from (3.3) we obtain for fixed $(|r| < 1, Re(\eta) \geq 0, \gamma \geq 0)$ or $(|r| \leq 1, Re(\eta) > 0, \gamma \geq 0)$ and $Re(\phi) = 0$,

$$Z(r, \phi, \eta, \gamma)K^{+}(r, \phi, \eta) = \frac{P^{\gamma}(\phi)\left[K^{-}(r, \phi, \eta)\right]^{-1}}{\prod_{i=1}^{k}(\phi + w_i)^{\gamma}} + V(r, \phi, \eta, \gamma)\left[K^{-}(r, \phi, \eta)\right]^{-1}. \tag{3.6}$$

In the complex ϕ plane, the left-hand side of (3.6) satisfies conditions A^{+} and the second term of the right-hand side satisfies conditions A^{-}. Suppose we can decompose the first term of the right hand side of (3.6) into two functions C^{+} and C^{-} such that for $Re(\phi) = 0$,

$$\frac{P^{\gamma}(\phi)}{\prod_{i=1}^{k}(\phi + w_i)^{\gamma}}\left[K^{-}(r, \phi, \eta)\right]^{-1} = C^{+}(r, \phi, \eta, \gamma) + C^{-}(r, \phi, \eta, \gamma), \tag{3.7}$$

where $C^+(r,\phi,\eta,\gamma)$ satisfies A^+ and $C^-(r,\phi,\eta,\gamma)$ satisfies A^-. We then have the following solution of (3.3).

Theorem 3.3.2
For $(|r|<1, Re(\phi)\geq 0, Re(\eta)\geq 0, \gamma\geq 0)$, *or* $(|r|\leq 1, Re(\phi)\geq 0, Re(\eta)>0, \gamma\geq 0)$, *or* $(|r|\leq 1, Re(\phi)>0, Re(\eta)\geq 0, \gamma\geq 0)$,*we have*

$$Z(r,\phi,\eta,\gamma) = \left[C^+(r,\phi,\eta,\gamma) + C^-(r,0,\eta,\gamma)\right]\left[K^+(r,\phi,\eta)\right]^{-1}. \tag{3.8}$$

Proof. From (3.6) and (3.7) we have

$$\begin{aligned} Z(r,\phi,\eta,\gamma)K^+(r,\phi,\eta) - C^+(r,\phi,\eta,\gamma) =& C^-(r,\phi,\eta,\gamma) \\ &+ V(r,\phi,\eta,\gamma)\left[K^-(r,\phi,\eta)\right]^{-1}. \end{aligned} \tag{3.9}$$

The left-hand side of (3.9) satisfies A^+ and the right-hand side satisfies A^-. By analytic continuation in the complex ϕ plane, we can define an entire function which is equal to the left-hand side for $Re(\phi)\geq 0$ and equal to the right-hand side for $Re(\phi)\leq 0$. This entire function is bounded, and hence by Liouville's theorem, it is a constant. So, for $Re(\phi)\geq 0$

$$\begin{aligned} Z(r,\phi,\eta,\gamma)K^+(r,\phi,\eta) - C^+(r,\phi,\eta,\gamma) &= Z(r,0,\eta,\gamma)K^+(r,0,\eta) - C^+(r,0,\eta,\gamma) \\ &= C^-(r,0,\eta,\gamma) + 0, \end{aligned} \tag{3.10}$$

with $(|r|\leq 1, Re(\eta)>0, \gamma\geq 0)$ or $(|r|<1, Re(\eta)\geq 0, \gamma\geq 0)$, which proves the theorem. ∎

If $\rho<1$ and both

$$K^+(1,\phi,0) = \lim_{r\uparrow 1} K^+(r,\phi,0) \text{ and } K^-(1,\phi,0) = \lim_{r\uparrow 1} K^-(r,\phi,0)$$

exist for $Re(\phi)\geq 0$, then from (3.8), in using Abel's theorem, the Laplace-Stieltjes transform of the steady-state waiting time distribution for $Re(\phi)\geq 0$ is given by

$$\begin{aligned} Z(\phi) &= \lim_{r\uparrow 1}(1-r)Z(r,\phi,0,\gamma) \\ &= \left[\lim_{r\uparrow 1}(1-r)\left[C^+(r,\phi,0,\gamma) + C^-(r,0,0,\gamma)\right]\right]\left[K^+(1,\phi,0)\right]^{-1}. \end{aligned} \tag{3.11}$$

The explicit expression for $Z(\phi)$ can then be found once we have explicit expressions for $K^+(r,\phi,\eta), K^-(r,\phi,\eta), C^+(r,\phi,\eta,\gamma)$, and $C^-(r,\phi,\eta,\gamma)$.

3.4 The distribution of the virtual waiting time

Let the number of arrivals in the interval $(0,t]$ be denoted by

$$N_t = \sup\{n=1,2,\ldots \mid T_n \leq t\}$$

and let

$$U_t = W_{N_t} + B_{N_t} - (t - T_{N_t}).$$

Then the virtual waiting time V_t is given by $V_t = U_t^+$. Notice that the sample paths of V_t are right-continuous. By the law of total probability we have for $Re(\phi) \geq 0, \gamma \geq 0$,

$$\begin{aligned} E\left[e^{-\phi U_t}|C_0=\gamma\right] &= E\left[e^{-\phi\left[W_{N_t}+B_{N_t}-(t-T_{N_t})\right]}|C_0=\gamma\right] \\ &= \sum_{n=0}^{\infty} E\left[e^{-\phi\left(W_{N_t}+B_{N_t}-(t-T_{N_t})\right)}\mathbf{1}(T_n \leq t < T_{n+1})|C_0=\gamma\right] \\ &= \sum_{n=0}^{\infty}\int_0^t e^{+\phi(t-u)}\{1-F(t-u)\}E\left(e^{-\phi B_n}\right) \\ &\qquad .d_u E\left[e^{-\phi W_n}\mathbf{1}(T_n \leq u)|C_0=\gamma\right] \\ &= B(\phi)\sum_{n=0}^{\infty}\int_0^t e^{+\phi(t-u)}\{1-F(t-u)\} \\ &\qquad .d_u E\left[e^{-\phi W_n}\mathbf{1}(T_n \leq u)|C_0=\gamma\right], \end{aligned} \tag{3.12}$$

where $\mathbf{1}(A)$ denotes the indicator function of the event A.

Hence, for $Re(\eta) > Re(\phi) \geq 0, \gamma \geq 0$, we find

$$\begin{aligned} &\int_0^{\infty} e^{-\eta t}E\left[e^{-\phi U_t}|C_0=\gamma\right]dt \\ =&\frac{B(\phi)-A(\eta-\phi)B(\phi)}{\eta-\phi}\sum_{n=0}^{\infty}E\left[e^{-\phi W_n-\eta T_n}|C_0=\gamma\right] \\ =&\frac{B(\phi)-A(\eta-\phi)B(\phi)}{\eta-\phi}Z(1,\phi,\eta,\gamma). \end{aligned} \tag{3.13}$$

Then, by using identity (3.4), we have for $Re(\eta) > 0, Re(\phi) = 0, \gamma \geq 0$,

$$\begin{aligned} Z^*(\phi,\eta,\gamma) = &\int_0^{\infty} e^{-\eta t}E\left[e^{-\phi V_t}|C_0=\gamma\right]dt \\ =&\frac{B(\phi)-A(\eta-\phi)B(\phi)}{\eta-\phi}Z(1,\phi,\eta,\gamma)+\frac{1}{\eta} \\ &-\int_0^{\infty} e^{-\eta t}E\left[e^{-\phi U_t^-}|C_0=\gamma\right]dt. \end{aligned} \tag{3.14}$$

We now decompose the term $\frac{B(\phi)-A(\eta-\phi)B(\phi)}{\eta-\phi}Z(1,\phi,\eta,\gamma)$, i.e. we determine two functions $D^+(\phi,\eta,\gamma)$ and $D^-(\phi,\eta,\gamma)$ such that for $Re(\phi) = 0$

$$\frac{B(\phi)-A(\eta-\phi)B(\phi)}{\eta-\phi}Z(1,\phi,\eta,\gamma) = D^+(\phi,\eta,\gamma) + D^-(\phi,\eta,\gamma), \tag{3.15}$$

where in the complex ϕ plane, $D^+(\phi,\eta,\gamma)$ satisfies A^+ and $D^-(\phi,\eta,\gamma)$ satisfies A^-. For this purpose, we first notice from equation (3.3), that for $Re(\phi)\geq 0$, $Re(\eta)>0, \gamma\geq 0$,

$$\begin{aligned}\frac{B(\phi)-A(\eta-\phi)B(\phi)}{\eta-\phi}Z(1,\phi,\eta,\gamma) &= \frac{B(\phi)Z(1,\phi,\eta,\gamma)}{\eta-\phi}-\frac{Z(1,\phi,\eta,\gamma)}{\eta-\phi}\\ &\quad+\frac{P(\phi)^\gamma}{(\eta-\phi)\prod_{i=1}^k(\phi+w_i)^\gamma}+\frac{V(1,\phi,\eta,\gamma)}{(\eta-\phi)}.\end{aligned} \tag{3.16}$$

By defining the function

$$F(\phi,\eta,\gamma)=B(\phi)Z(1,\phi,\eta,\gamma)-Z(1,\phi,\eta,\gamma)+\frac{P^\gamma(\phi)}{\prod_{i=1}^k(\phi+w_i)^\gamma}, \tag{3.17}$$

for $Re(\phi)\geq 0, Re(\eta)>0, \gamma\geq 0$, we can choose

$$D^+(\phi,\eta,\gamma)=\frac{F(\phi,\eta,\gamma)-F(\eta,\eta,\gamma)}{(\eta-\phi)} \tag{3.18}$$

and

$$D^-(\phi,\eta,\gamma)=\frac{F(\eta,\eta,\gamma)}{(\eta-\phi)}+\frac{V(1,\phi,\eta,\gamma)}{(\eta-\phi)} \tag{3.19}$$

With this decomposition, we have the following result.

Theorem 3.4.1
For $Re(\eta)\geq 0, Re(\phi)\geq 0, \gamma\geq 0$,

$$Z^*(\phi,\eta,\gamma)=D^+(\phi,\eta,\gamma)+\frac{F(\eta,\eta,\gamma)}{\eta}. \tag{3.20}$$

Proof. With the decomposition (3.15) we can rewrite (3.14) as

$$\begin{aligned}Z^*(\phi,\eta,\gamma)-D^+(\phi,\eta,\gamma) &= D^-(\phi,\eta,\gamma)+\frac{1}{\eta}\\ &\quad-\int_0^\infty e^{-\eta t}E\left[e^{-\phi U_t^-}|C_0=\gamma\right]dt,\end{aligned} \tag{3.21}$$

where in the complex ϕ plane the left-hand side of (3.21) satisfies A^+ and the right-hand side satisfies A^-. By analytic continuation, we can define an entire function which is equal to the left-hand side for $Re(\phi)\geq 0$ and equal to the right-hand side for $Re(\phi)<0$. This entire function is bounded, and hence by Liouville's theorem, it is a constant. Therefore,

$$\begin{aligned}Z^*(\phi,\eta,\gamma)-D^+(\phi,\eta,\gamma) &= Z^*(0,\eta,\gamma)-D^+(0,\eta,\gamma)\\ &= D^-(0,\eta,\gamma)+\frac{1}{\eta}-\frac{1}{\eta}\\ &= \frac{F(\eta,\eta,\gamma)}{\eta},\end{aligned} \tag{3.22}$$

and we get (3.20).

■

If $\rho < 1$ and if the distribution function of the inter-arrival times F is non-lattice, then the steady-state virtual waiting time distribution exists. Let

$$Z^*(\phi) = \lim_{t\to\infty} E\left[e^{-\phi V_t} | C_0 = \gamma\right].$$

From Abel's theorem for Laplace transforms, we obtain

$$\begin{aligned}
Z^*(\phi) &= \lim_{\eta\downarrow 0} \eta Z^*(\phi,\eta,\gamma) \\
&= \lim_{\eta\downarrow 0} \eta \left[D^+(\phi,\eta,\gamma) + \frac{F(\eta,\eta,\gamma)}{\eta}\right] \\
&= \lim_{\eta\downarrow 0} \frac{\eta}{(\eta-\phi)}\left[B(\phi)Z(1,\phi,\eta,\gamma) - Z(1,\phi,\eta,\gamma) + \frac{P^\gamma(\phi)}{\prod_{i=1}^k(\phi+w_i)^\gamma}\right] \\
&\quad - \lim_{\eta\downarrow 0} \frac{\phi}{(\eta-\phi)} F(\eta,\eta,\gamma) \\
&= \frac{1-B(\phi)}{\phi} \lim_{\eta\downarrow 0} \eta Z(1,\phi,\eta,\gamma) - \lim_{\eta\downarrow 0} \frac{1-B(\eta)}{\eta} \cdot \lim_{\eta\downarrow 0} \eta Z(1,\eta,\eta,\gamma) \\
&= 1-\rho+\rho\frac{1-B(\phi)}{\beta\phi} Z(\phi),
\end{aligned} \tag{3.23}$$

a well known relation for the $GI/G/1$ queue that relates the L-S transforms of the probability distributions of the virtual and actual waiting time.

3.5 Number of customers at arrival epochs

Let C_n be the number of customers in the system at T_n^-, i.e. just before the arrival of the nth customer. It is clear that

$$\{C_0 \le j\} = \begin{cases} \text{impossible event} & , j = 0,1,\ldots,C_0-1 \\ \Omega & , j = C_0, C_0+1,\ldots, \end{cases}$$

where Ω is the sure event. Furthermore, for $n = 1,2,\ldots,j$,

$$\{C_n \le j\} = \begin{cases} \left\{\sum_{i=1}^{C_0-(j-n)} X_i < T_n\right\} & , j = 1,2,\ldots,C_0 \\[2ex] \Omega & , j = C_0+1, C_0+2,\ldots, \& \\ & \quad n = 1,2,\ldots,j-C_0 \\[2ex] \left\{\sum_{i=1}^{C_0-(j-n)} X_i < T_n\right\} & , j = C_0+1, C_0+2,\ldots, \& \\ & \quad n = j-C_0+1, j-C_0+2,\ldots,j \end{cases}$$

and

$$\{C_{n+j+1} \leq j\} = \{T_n + W_n + B_n < T_{n+j+1}\} \quad n = 0, 1, \ldots.$$

Theorem 3.5.1
For $|r| < 1, |s| < 1, \gamma \geq 0,$

$$\begin{aligned} U(r,s,\gamma) &= \sum_{n=0}^{\infty} r^n E\left[s^{C_n} | C_0 = \gamma\right] \\ &= \frac{rs^{\gamma+1}}{(1-rs)} + s^{\gamma} \\ &\quad + \frac{(1-s)}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \left[\frac{srA(-\xi)P(\xi)\left[P^{\gamma}(\xi) - s^{\gamma}\prod_{i=1}^{k}(\xi+w_i)^{\gamma}\right]}{X_1(r,\xi)X_2(s,\xi)\prod_{i=1}^{k}(\xi+w_i)^{\gamma}} \right] \\ &\quad + (1-s)\frac{r}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}(1-rsA(-\xi))^{-1}A(-\xi)B(\xi)Z(r,\xi,0,\gamma), \end{aligned} \tag{3.24}$$

where

$$X_1(r,s,\xi) = 1 - srA(-\xi),$$

and

$$X_2(s,\xi) = P(\xi) - s\prod_{i=1}^{k}(\xi + w_i).$$

Proof. For $|r| < 1, |s| < 1, \gamma \geq 0,$

$$\begin{aligned} &\sum_{n=0}^{\infty} r^n E\left[s^{C_n}|C_0\gamma\right] \\ &= (1-s)\sum_{n=0}^{\infty} r^n \sum_{j=0}^{\infty} s^j P(C_n \leq j) \\ &= (1-s)\sum_{j=\gamma}^{\infty} s^j + (1-s)\sum_{j=\gamma+1}^{\infty}\sum_{n=1}^{j-\gamma} r^n s^j \\ &\quad + (1-s)\sum_{n=1}^{\infty}\sum_{j=n}^{n+\gamma-1} r^n s^j P\left(\sum_{i=1}^{\gamma-(j-n)} X_i < T_n\right) \\ &\quad + (1-s)\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} r^{n+j+1}s^j P(T_n + W_n + B_n < T_{n+j+1}|C_0 = \gamma). \end{aligned} \tag{3.25}$$

We use the identity 2.1.3, that is

$$\mathbf{1}(x<0) + \frac{1}{2}\mathbf{1}(x=0) = \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}e^{-\xi x}$$

to obtain

$$P\left(\sum_{i=1}^{\gamma-(j-n)} X_i < T_n\right) = \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} E\left[e^{-\xi(\sum_{i=1}^{\gamma-(j-n)} X_i - T_n)}\right] = \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} E\left[e^{-\xi(\sum_{i=1}^{\gamma-(j-n)} X_i)}\right] A^n(-\xi) \tag{3.26}$$

and

$$\begin{aligned} &P(T_n + W_n + B_n < T_{n+j+1}|C_0 = \gamma) \\ =&\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} E\left[e^{-\xi(T_n+W_n+B_n-T_{n+j+1})}|C_0 = \gamma\right] \\ =&\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} E\left[e^{-\xi W_n}|C_0 = \gamma\right] B(\xi)A^{j+1}(-\xi), \end{aligned} \tag{3.27}$$

taking into account the independence assumptions. By substituting (3.26) and (3.27) into (3.25) we obtain for $|r| < 1, |s| < 1, \gamma \geq 0$,

$$\begin{aligned} &\sum_{n=1}^{\infty} r^n E\left[s^{C_n}|C_0 = \gamma\right] \\ =&(1-s)\sum_{n=1}^{\infty}\sum_{j=n}^{n+\gamma-1} r^n s^j \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} E\left[e^{-\xi(\sum_{i=1}^{\gamma-(j-n)} X_i)}\right] A^n(-\xi) \\ &+ \frac{rs^{\gamma+1}}{(1-rs)} + s^\gamma \\ &+ (1-s)\frac{1}{2\pi i} r \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}(1 - rsA(-\xi))^{-1}A(-\xi)B(\xi)Z(r,\xi,0,\gamma). \end{aligned} \tag{3.28}$$

Since the r.v.'s $X_i, \quad i = 1, \ldots, \gamma$, are independent the first term of the right-hand side of (3.28) can be written as

$$\begin{aligned} &(1-s)\sum_{n=1}^{\infty}\sum_{j=n}^{n+\gamma-1} r^n s^j \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} E\left[e^{-\xi X_1}\right]^{(\gamma-j+n)} A^n(-\xi) \\ =&\frac{(1-s)}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \mathcal{P}^\gamma(\xi) \sum_{n=1}^{\infty}\sum_{j=n}^{n+\gamma-1} (rA(-\xi)\mathcal{P}(\xi))^n \left(\frac{s}{\mathcal{P}(\xi)}\right)^j \\ =&\frac{(1-s)}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \mathcal{P}^\gamma(\xi) \frac{srA(-\xi)}{1-srA(-\xi)} \sum_{j=0}^{\gamma-1} \left(\frac{s}{\mathcal{P}(\xi)}\right)^j \\ =&\frac{(1-s)}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \frac{rA(-\xi)\mathcal{P}(\xi)s\left[\mathcal{P}^\gamma(\xi) - s^\gamma\right]}{(1-srA(-\xi))\left(\mathcal{P}(\xi) - s\right)} \\ =&\frac{(1-s)}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \frac{srA(-\xi)P(\xi)[P^\gamma(\xi) - s^\gamma \prod_{i=1}^{k}(\xi + w_i)^\gamma]}{(1-srA(-\xi))(P(\xi) - s\prod_{i=1}^{k}(\xi + w_i))\prod_{i=1}^{k}(\xi + w_i)^\gamma}. \end{aligned} \tag{3.29}$$

By inserting (3.29) into (3.28), the proof is completed.

■

The generating function of the nth moment of number of customers at arrival epochs can be derived from (3.24). For the first moment one obtains the following relation. For $|r| < 1, \gamma \geq 0$,

$$\begin{aligned} U(r,\gamma) &= \sum_{n=0}^{\infty} r^n E\left[C_n | C_0 = \gamma\right] \\ &= \frac{\gamma}{1-r} + \frac{r}{(1-r)^2} \\ &\quad - \frac{r}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\frac{A(-\xi)}{1-rA(-\xi)}\sum_{j=1}^{\gamma}\mathcal{P}^j(\xi) \\ &\quad - \frac{r}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}[1-rA(-\xi)]^{-1}A(-\xi)B(\xi)Z(r,\xi,0,\gamma). \end{aligned} \tag{3.30}$$

The first integral can be evaluated using contour integration leading to the following theorem.

Theorem 3.5.2
For $r < 1$, $\gamma \geq 0$, and $w_i \neq w_j$ for $i \neq j$

$$\begin{aligned} U(r,\gamma) =& \gamma + \frac{r}{(1-r)^2} \\ & - r\sum_{j=1}^{\gamma}\frac{1}{(j-1)!}\sum_{i=1}^{k}\frac{d^j}{d\xi^j}\left[\frac{A(-\xi)P(\xi)^j}{\xi(1-rA(-\xi))\prod_{n=1,n\neq i}^{k}(\xi+w_n)^j}\right]_{\xi=-w_i} \\ & - \frac{r}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}\left[1-rA(-\xi)\right]^{-1}A(-\xi)B(\xi)Z(r,\xi,0,\gamma). \end{aligned} \tag{3.31}$$

Proof. We want to evaluate the first integral in (3.29) using Cauchy's residue theorem. Consider a closed contour consisting of the line segment $[-iR+\delta, iR+\delta], \delta > 0$, parallel to the imaginary axis in the complex ξ plane and a left semi-circle Γ_R closing the contour. The integrand has a simple pole at $\xi = 0$ and in view of (3.1) has poles in $\xi = -w_i, i = 1, 2, \ldots, k$, each of which occurs with orders $j = 1, 2, \ldots, \gamma$, since $w_i \neq w_j (i \neq j)$ by assumption. Observe that $1 - rA(-\xi) \neq 0$ within the closed contour.

The residue at $\xi = 0$ equals $\frac{\gamma}{1-r}$, since $\mathcal{P}(0) = 1$ and $A(0) = 1$. The residue at $\xi = -w_i$, a_{ij} say, corresponding to $\mathcal{P}(\xi)^j$, cf. (3.1), equals

$$a_{ij} = \frac{1}{(j-1)!}\frac{d^j}{d\xi^j}\left[\frac{A(-\xi)P(\xi)^j}{\xi(1-rA(-\xi))\prod_{\substack{n=1\\n\neq i}}^{k}(\xi+w_n)^j}\right]_{\xi=-w_i}.$$

Hence, letting $f(\xi)$ denote the integrand of the integral, we obtain

$$\int_{-i\infty+0}^{i\infty+0} f(\xi)d\xi + \int_{\Gamma_R} f(\xi)d\xi = \frac{r\gamma}{1-r} + r\sum_{j=1}^{\gamma}\sum_{i=1}^{k}a_{ij}$$

Now observe that

$$|\sum_{j=1}^{\gamma} \mathcal{P}(\xi)| = O\left(\frac{1}{|\xi|}\right) \text{ as } |\xi| \to \infty$$

and $|A(-\xi)| \leq M$ for $Re(\xi) < \delta$.

Hence

$$\lim_{R\to\infty} \int_{\Gamma_R} f(\xi) d\xi = 0$$

and the assertion follows after some simple calculations.

■

Theorem 3.5.2 will be used in section 3.7 where an explicit expression for the last integral in the right-hand side of (3.31) is derived.

The process $\{C_n, n = 0, 1, \ldots\}$ is regenerative with the same regeneration points as the process $\{W_n, n = 0, 1, \ldots\}$, because the events $\{C_n = 0\}$ and $\{W_n = 0\}$ are identical. Therefore $\{C_n\}$ converges weakly to a random variable C iff $\rho < 1$. By using Abel's theorem and (3.11), we have for $\rho < 1$,

$$E\left[s^C\right] = \frac{(1-s)}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} (1 - sA(-\xi))^{-1} A(-\xi) B(\xi) Z(\xi),$$

and

$$E[C] = -\frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} (1 - A(-\xi))^{-1} A(-\xi) B(\xi) Z(\xi). \tag{3.32}$$

In section 3.7, where we have an explicit expression for $A(-\xi)$ or $B(\xi)$, the integral can be calculated, and we will get a closed-form expression for $E[C]$.

3.6 Number of customers in continuous time

Let C_t^* be the number of customers at time t. The process (C_t^*) is defined to be left-continuous. Then by partitioning the event $\{C_t^* \leq j\}$ with respect to the number of customers that enter the system in $(0, t]$, we have for $j = 0, 1, \ldots, C_0$,

$$\begin{aligned}
\{C_t^* \leq j\} = & \bigcup_{n=C_0-j+1}^{C_0} \left\{ \sum_{i=1}^{n} X_i < t, N_t = j - C_0 - 1 + n \right\} \cup \\
& \bigcup_{n=C_0+1}^{\infty} \{T_{n-C_0-1} + W_{n-C_0-1} + B_{n-C_0-1} < t, N_t = j - C_0 - 1 + n\} \\
= & \bigcup_{n=C_0-j+1}^{C_0} \left\{ \sum_{i=1}^{n} X_i < t, \quad T_{j-C_0-1+n} \leq t < T_{j-C_0+n} \right\} \cup \\
& \bigcup_{n=0}^{\infty} \{T_n + W_n + B_n < t, \quad T_{j+n} \leq t < T_{j+n+1}\},
\end{aligned}$$

and for $j = C_0 + 1, C_0 + 2, \ldots,$

$$\begin{aligned}
\{C_t^* \le j\} = &\{N_t \le j - C_0 - 1\} \cup \\
&\bigcup_{n=1}^{C_0} \left\{ \sum_{i=1}^{n} X_i < t, N_t = j - C_0 - 1 + n \right\} \cup \\
&\bigcup_{n=C_0+1}^{\infty} \{T_{n-C_0-1} + W_{n-C_0-1} + B_{n-C_0-1} < t, N_t = j - C_0 - 1 + n\} \\
= &\{t < T_{j-C_0}\} \\
&\cup \bigcup_{n=1}^{C_0} \left\{ \sum_{i=1}^{n} X_i < t, T_{j-C_0-1+n} \le t < T_{j-C_0+n} \right\} \\
&\cup \bigcup_{n=0}^{\infty} \{T_n + W_n + B_n < t, T_{j+n} \le t < T_{j+n+1}\}.
\end{aligned}$$

This leads to

$$\begin{aligned}
&\int_0^\infty e^{-\eta t} E\left[s^{C_t^*} | C_0 = \gamma\right] dt \\
&= (1-s) \sum_{j=0}^{\infty} s^j \int_0^\infty e^{-\eta t} P\left(C_t^* \le j | C_0 = \gamma\right) dt \\
&= (1-s) \sum_{j=0}^{\gamma} s^j \sum_{n=\gamma+1-j}^{\gamma} \int_0^\infty e^{-\eta t} P\left(\sum_{i=1}^{n} X_i < t, T_{j-\gamma-1+n} \le t < T_{j-\gamma+n} \right) dt \\
&\quad + (1-s) \sum_{j=0}^{\gamma} s^j \sum_{n=0}^{\infty} \int_0^\infty e^{-\eta t} P\left(T_n + W_n + B_n < t, T_{j+n} \le t < T_{j+n+1}\right) dt \\
&\quad + (1-s) \sum_{j=0}^{\infty} s^{j+\gamma+1} \int_0^\infty e^{-\eta t} P\left(t < T_{j+1}\right) dt \\
&\quad + (1-s) \sum_{j=0}^{\infty} s^{j+\gamma+1} \sum_{n=1}^{\gamma} \int_0^\infty e^{-\eta t} P\left(\sum_{i=1}^{n} X_i < t, T_{j+n} \le t < T_{j+n+1} \right) dt \\
&\quad + (1-s) \sum_{j=\gamma+1}^{\infty} s^j \sum_{n=0}^{\infty} \int_0^\infty e^{-\eta t} P\left(T_n + W_n + B_n < t, T_{j+n} \le t < T_{j+n+1}\right) dt,
\end{aligned}$$

for $|s| < 1, \gamma \ge 0$.

Upon combining the first and fourth term and the second and fifth term one obtains for

$|s| < 1, \gamma \geq 0,$

$$\begin{aligned}
&\int_0^\infty e^{-\eta t} E[s^{C_t^*}|C_0 = \gamma] dt \\
&= (1-s)\Bigg[\sum_{j=0}^{\infty} s^{j+\gamma+1} \int_0^\infty e^{-\eta t} P(T_{j+1} > t) dt \\
&\quad + \sum_{j=0}^{\infty} s^j \sum_{n=0}^{\infty} \int_0^\infty e^{-\eta t} P(T_n + W_n + B_n < t, T_{j+n} \leq t < T_{j+n+1}|C_0 = \gamma) dt \\
&\quad + \sum_{n=1}^{\gamma} \sum_{j=0}^{\infty} s^{j+\gamma-n+1} \int_0^\infty e^{-\eta t} P(\sum_{i=1}^{n} X_i < t, T_j \leq t < T_{j+1}) dt \Bigg].
\end{aligned} \tag{3.33}$$

By using the identity 2.1.3 on page 10 it follows that for $Re(\eta) > Re(\xi) > 0$, $\gamma \geq 0$,

$$\begin{aligned}
&\int_0^\infty e^{-\eta t} E\left[s^{C_t^*}|C_0 = \gamma\right] dt \\
=&\frac{s^{\gamma+1}}{\eta} - (1-s)\frac{s^{\gamma+1}}{\eta}\frac{A(\eta)}{(1-sA(\eta))} \\
&+ (1-s)\sum_{j=0}^{\infty} s^j \sum_{u=0}^{\infty} \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \int_0^\infty e^{-\eta t} \int_{u=0}^{t} \{1 - F(t-u)\} \\
&d_u E[e^{-\xi(T_n - t + W_n + B_n)} \mathbf{1}(T_{n+j} \leq u)|C_0 = \gamma] dt \\
&+ (1-s)\sum_{n=1}^{\gamma}\sum_{j=0}^{\infty} s^{j+\gamma-1+n} \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \int_0^\infty e^{-\eta t} \int_{u=0}^{t} \{1 - F(t-u)\} \\
&d_u E[e^{-\xi(\sum_{i=1}^{n} X_i - t)} \mathbf{1}(T_j \leq u)] dt \\
=&\frac{s^{\gamma+1}}{\eta} - (1-s)\frac{s^{\gamma+1}}{\eta}\frac{A(\eta)}{1-sA(\eta)} \\
&+ (1-s)\sum_{j=0}^{\infty} s^j \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\frac{1-A(\eta-\xi)}{\eta-\xi} B(\xi)\int_0^\infty e^{-(\eta-\xi)u} d_u E \\
&(e^{-\xi(T_n + W_n)} \mathbf{1}(T_{n+j} \leq u)|C_0 = \gamma) \\
&+ \frac{(1-s)}{2\pi i}\sum_{n=1}^{\gamma}\sum_{j=0}^{\infty} s^{j+\gamma-1+n} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\frac{1-A(\eta-\xi)}{\eta-\xi} E(e^{-\xi\sum_{i=1}^{n} X_i}) A^j(\eta-\xi) \\
=& \frac{s^{\gamma+1}}{\eta} - (1-s)\frac{s^{\gamma+1}}{\eta}\frac{A(\eta)}{1-sA(\eta)} \\
&+ \frac{(1-s)}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\frac{1-A(\eta-\xi)}{\eta-\xi} B(\xi)\frac{Z(1,\xi,\eta,\gamma)}{1-sA(\eta-\xi)} \\
&+ \frac{(1-s)}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\cdot\frac{1-A(\eta-\xi)}{\eta-\xi}\cdot\frac{s^{\gamma}\mathcal{P}(\xi)}{1-sA(\eta-\xi)}\cdot\frac{1-s^{\gamma}\mathcal{P}(\xi)^{\gamma}}{1-s\mathcal{P}(\xi)}.
\end{aligned} \tag{3.34}$$

The Laplace transform of the nth moments of C_t^* can be derived from (3.34). For the first moment we obtain, with $Re(\eta) > Re(\xi) > 0$, $\gamma \geq 0$,

$$\begin{aligned}\int_0^\infty e^{-\eta t} E\left[C_t^* | C_0 = \gamma\right] dt = &\frac{\gamma+1}{\eta} + \frac{A(\eta)}{\eta(1-A(\eta))}\\ &- \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\frac{B(\xi)Z(1,\xi,\eta,\gamma)}{(\eta-\xi)}\\ &- \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\frac{\mathcal{P}(\xi)}{\eta-\xi}\cdot\frac{1-\mathcal{P}(\xi)^\gamma}{1-\mathcal{P}(\xi)}.\end{aligned} \tag{3.35}$$

Since $B(\xi), \mathcal{P}(\xi)$ and $Z(1,\xi,\eta,\gamma)$ are regular functions in the right half-plane $Re(\xi) > 0$ and, moreover $|\mathcal{P}(\xi)| < 1$ for $Re(\xi) > 0$, and noting also that the integrands tends to zero sufficiently fast for $|\xi| \to \infty$ if $Re(\xi) > 0$, it follows from contour integration that for $Re(\eta) > 0$, $\gamma \geq 0$,

$$\begin{aligned}\int_0^\infty e^{-\eta t} E\left[C_t^* | C_0 = \gamma\right] dt = &\frac{\gamma+1}{\eta} + \frac{A(\eta)}{\eta(1-A(\eta))}\\ &- \frac{B(\eta)Z(1,\eta,\eta,\gamma)}{\eta} - \frac{1}{\eta}\mathcal{P}(\eta)\frac{1-\mathcal{P}(\eta)^\gamma}{1-\mathcal{P}(\eta)}.\end{aligned} \tag{3.36}$$

The process $\{C_t^*, t \geq 0\}$ is regenerative with the same regeneration epochs as $\{V_t, t \geq 0\}$. Consequently, (C_t^*) converges for $t \to \infty$ to a stationary random variable C^* iff $\rho < 1$ and the interarrival time distribution is non-lattice. We find from (3.34) for $|s| < 1$,

$$E\left[s^{C^*}\right] = \frac{(1-s)}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\lim_{\eta\downarrow 0}\frac{\eta B(\xi)(1-A(\eta-\xi))Z(1,\xi,\eta,\gamma)}{(\eta-\xi)(1-sA(\eta-\xi))}. \tag{3.37}$$

Since we have assumed that there exists a $\delta > 0$ such that $B(\phi)$ can be continued analytically into the region $Re(\delta) > -\delta$, then the same applies to $Z(1,\phi,\eta,\gamma)$. Notice that

$$\begin{aligned}&\lim_{\eta\downarrow 0}\frac{(1-A(\eta))}{\eta(1-sA(\eta))}\lim_{\eta\downarrow 0}\eta Z(1,0,\eta,\gamma)\\ &= \lim_{\eta\downarrow 0}\frac{-A'(\eta)}{(1-sA(\eta))-s\eta A'(\eta)}\lim_{\eta\downarrow 0}\eta Z(1,0,\eta,\gamma)\\ &= \frac{\alpha}{(1-s)}\lim_{\eta\downarrow 0}\eta Z(1,0,\eta,\gamma).\end{aligned} \tag{3.38}$$

Now we have for $|s| < 1$,

$$\begin{aligned}E\left[s^{C^*}\right] = &\,\alpha\lim_{\eta\downarrow 0}\eta Z(1,0,\eta,\gamma)\\ &+ \frac{(1-s)}{2\pi i}\int_{-i\infty-0}^{i\infty-0}\frac{d\xi}{\xi}\lim_{\eta\downarrow 0}\frac{\eta B(\xi)(1-A(\eta-\xi))Z(1,\xi,\eta,\gamma)}{(\eta-\xi)(1-sA(\eta-\xi))},\end{aligned} \tag{3.39}$$

and it yields

$$E[C^*] = -\frac{1}{2\pi i}\int_{-i\infty-0}^{i\infty-0}\frac{d\xi}{\xi}\frac{B(\xi)}{\xi}\lim_{\eta\downarrow 0}\eta Z(1,\xi,\eta,\gamma). \tag{3.40}$$

Here we also need the expression for $\lim_{\eta\downarrow 0}\eta Z(1,\xi,\eta)$ to analyze the integral in (3.40). For this reason, the further study will be done in section 3.7.

3.7 The systems $GI/K_n/1$ and $K_m/G/1$

In this section, we study the two special cases of $GI/G/1$ in which either the inter-arrival time distribution or the service time distribution has a rational Laplace-Stieltjes transform. For these cases, the factorization of (3.5) can be done easily, and it yields explicit expressions for (3.11),(3.23), (3.32), and (3.40).

3.7.1 The system $GI/K_n/1$

The Laplace-Stieltjes transform of the service time for this model has the form

$$B(\phi) = \frac{B_1(\phi)}{\prod_{i=1}^{n}(\phi + \mu_i)},$$

where $Re(\mu_i) > 0$, $i = 1, 2, \ldots, n$, and $B_1(\phi)$ is a polynomial of degree $(n-1)$ or less. Now we have

$$1 - rA(\eta - \phi)B(\phi) = \frac{\prod_{i=1}^{n}(\phi + \mu_i) - rA(\eta - \phi)B_1(\phi)}{\prod_{i=1}^{n}(\phi + \mu_i)}. \tag{3.41}$$

For $\delta > 0$, consider the contour $C^-_{\delta,R}$ in the complex ϕ plane. For $|r| < 1$ and $Re(\eta) \geq 0$ or $|r| \leq 1$ and $Re(\eta) > 0$ with $\phi \in C^-_{\delta,R}$, then for R large enough

$$|rA(\eta - \phi)B_1(\phi)| < |\prod_{i=1}^{n}(\phi + \mu_i)| \text{ with } |\phi| = R, Re(\phi) < 0.$$

Moreover, since for $Re(\phi) = -\delta$ and $Re(\eta) \geq 0$

$$\begin{aligned} |rA(\eta - \phi)B(\phi)| &\leq |r|A(Re(\eta - \phi))B(Re(\phi)) \\ &\leq |r|A(\delta)B(-\delta). \end{aligned}$$

Since

$$A(\delta)B(-\delta) = 1 + \alpha(1-\rho)\delta + o(\delta), \delta \downarrow 0, \rho = \beta/\alpha$$

it follows that

$$|rA(\eta - \phi)B_1(\phi)| < |\prod_{i=1}^{n}(\phi + \mu_i)| \text{ with } Re(\phi) = -\delta$$

for $|r| < 1$ or $|r| = 1, \rho < 1$. Hence, by Rouché's theorem the function (3.40) has exactly n zeros $\lambda_i(r, \eta), i = 1, 2, \ldots, n$ in the left half-plane $Re(\phi) < 0$ if $(|r| < 1, Re(\eta) \geq 0)$ or $(|r| = 1, \rho < 1, Re(\eta) \geq 0)$. These zeros are continuous in r for $|r| \leq 1$, so that

$$\lim_{r \uparrow 1} \lambda_i(r, \eta) = \lambda_i(1, \eta).$$

It follows that

$$1 - rA(\eta - \phi)B(\phi) = K^+(r, \phi, \eta)K^-(r, \phi, \eta), \quad Re(\phi) = 0,$$

with

$$K^{-}(r,\phi,\eta)=\frac{\prod_{i=1}^{n}(\phi+\mu_i)-rA(\eta-\phi)B_1(\phi)}{\prod_{i=1}^{n}(\phi-\lambda_i(r,\eta))} \tag{3.42}$$

and

$$K^{+}(r,\phi,\eta)=\prod_{i=1}^{n}\frac{(\phi-\lambda_i(r,\eta))}{(\phi+\mu_i)}. \tag{3.43}$$

It is clear that $K^{+}(r,\phi,\eta)$ satisfies $\tilde{A}^{+}$ and $K^{-}(r,\phi,\eta)$ satisfies $\tilde{A}^{-}$. For the decomposition indicated in (3.7) we impose the following condition.

Condition 3.7.1
$-w_i$ and $\lambda_j(r,\eta)$, $\quad i=1,2,\ldots,k, \quad j=1,2,\ldots,n$, are all distinct.

We expand (3.2) into partial fractions

$$\frac{P^{\gamma}(\phi)}{\prod_{i=1}^{k}(\phi+w_i)^{\gamma}}=\sum_{i=1}^{k}\sum_{j=1}^{\gamma}\frac{q_{ij}}{(\phi+w_i)^j} \tag{3.44}$$

where

$$q_{ij}=\frac{1}{(\gamma-j)!}\frac{d^{\gamma-j}}{d\phi^{\gamma-j}}\left[\frac{P^{\gamma}(\phi)}{\prod_{n=1,n\neq i}^{k}(\phi+w_n)^{\gamma}}\right]_{\phi=-w_i} \tag{3.45}$$

Notice that $\sum_1^k\sum_1^{\gamma} q_{ij}/w_i^j=1$, since $\mathcal{P}(0)=1$.
Let

$$h_i^{(j)}(r,\eta)=\frac{1}{j!}\frac{d^j}{d\phi^j}\left[K^{-}(r,\phi,\eta)\right]^{-1}|_{\phi=-w_i} \tag{3.46}$$

where $h_i^{(0)}(r,\eta)=[K^{-}(r,-w_i,\eta)]^{-1}$.

To find the decomposition for $C^{+}(r,\phi,\eta,\gamma)$ and $C^{-}(r,\phi,\eta,\gamma)$, see (3.7), we now choose

$$C^{-}(r,\phi,\eta,\gamma)=\sum_{i=1}^{k}\sum_{j=1}^{\gamma}\frac{q_{ij}}{(\phi+w_i)^j}\left\{\left[K^{-}(r,\phi,\eta)\right]^{-1}-\sum_{l=0}^{j-1}h_i^{(l)}(r,\eta)(\phi+w_i)^l\right\} \tag{3.47}$$

and

$$C^{+}(r,\phi,\eta,\gamma)=\sum_{i=1}^{k}\sum_{j=1}^{\gamma}\sum_{l=0}^{j-1}q_{ij}\frac{h_i^{(l)}(r,\eta)}{(\phi+w_i)^{j-l}}. \tag{3.48}$$

Since $K^{-}(r,\phi,\eta)$ satisfies $\tilde{A}^{-}$ in the ϕ plane, it is readily seen that as a function of ϕ, $C^{+}(r,\phi,\eta,\gamma)$ satisfies A^{+} and $C^{-}(r,\phi,\eta,\gamma)$ satisfies A^{-}.

The actual waiting time

From (3.8), (3.43), (3.47) and (3.48) we have for ($|r| < 1, Re(\phi) \geq 0, Re(\eta) \geq 0, \gamma \geq 0,$), or ($|r| \leq 1, Re(\phi) \geq 0, Re(\eta) > 0, \gamma \geq 0$), or ($|r| \leq 1, Re(\phi) > 0, Re(\eta) \geq 0, \gamma \geq 0$),

$$\begin{aligned} Z(r,\phi,\eta,\gamma) &= \prod_{i=1}^{n} \frac{(\phi+\mu_i)}{(\phi-\lambda_i(r,\eta))} \sum_{j=1}^{k} \sum_{l=1}^{\gamma} \sum_{m=0}^{l-1} q_{jl} \frac{h_j^{(m)}(r,\eta)}{(\phi+w_j)^{l-m}} \\ &\quad + \prod_{i=1}^{n} \frac{(\phi+\mu_i)}{(\phi-\lambda_i(r,\eta))} C^-(r,0,\eta,\gamma). \end{aligned} \tag{3.49}$$

To find the distribution of actual waiting times W_n, we consider the function $Z(r,\phi,\eta,\gamma)$ for $\eta = 0$. This function is a rational function in ϕ, so that we can invert it analytically to obtain the generating function

$$G(r,x,\gamma) = \sum_{m=0}^{\infty} r^m P(W_m \leq x | C_0 = \gamma), \quad |r| < 1, x \geq 0, \gamma \geq 0. \tag{3.50}$$

For this inversion, let us define

$$p_i = \frac{\prod_{j=1}^{n} (\lambda_i(r,0)+\mu_j)}{\lambda_i(r,0) \prod_{j=1, j\neq i}^{n} (\lambda_i(r,0)-\lambda_j(r,0))}, \tag{3.51}$$

Then by using (A.3) and (3.51) we have for $|r| < 1, x \geq 0$,

$$\begin{aligned} G(r,x,\gamma) &= \frac{1}{(1-r)} + \sum_{i=1}^{n} p_i \left(C^-(r,0,0,\gamma) + C^+(r,\lambda_i(r,0),0,\gamma) \right) e^{\lambda_i(r,0)x} \\ &\quad + \sum_{i=1}^{k} \sum_{j=1}^{\gamma} \sum_{m=0}^{j-1} q_{ij}^{-1} h_i^{(m)}(r,\eta) \sum_{l=1}^{l-m} \frac{\Phi_l(-w_i) x^{l-m-1} e^{-w_j x}}{(j-m-l)!(l-1)!}, \end{aligned} \tag{3.52}$$

where

$$\Phi_l(\phi) = \frac{\partial^{l-1}}{\partial \phi^{l-1}} \left(\frac{1}{\phi} K^+(r,\phi,0) \right)^{-1}.$$

By a numerical inversion of (3.52), we get the distribution function $P(W_n \leq x)$.

For $\rho < 1$, we consider the steady-state distribution of the waiting times. For this purpose, let

$$\hat{\lambda}_i(1) = \lim_{r \uparrow 1} \hat{\lambda}_i(r) = \lim_{r \uparrow 1} \lambda_i(r,0), \quad i = 1, \ldots, n,$$

where the existence of the limit is discussed on page 32.

By definition we see that for $j = 1, 2, \ldots, k$,

$$\lim_{r \uparrow 1} (1-r) \left(K^-(r,-w_j,\eta) \right)^{-1} = 0.$$

That implies

$$\lim_{r\uparrow 1}(1-r)C^{+}(r,\phi,0,\gamma)=0.$$

Moreover, since $\sum_1^k\sum_1^\gamma q_{ij}/w_i^j=1$,

$$\lim_{r\uparrow 1}(1-r)C^{-}(r,0,0,\gamma)=\prod_{i=1}^{n}\frac{-\hat{\lambda}_i(1)}{\mu_i}.$$

Then from (3.11) and by using Abel's theorem we have

$$Z(\phi)=\lim_{m\to\infty}E\left[e^{-\phi W_m}|C_0=\gamma\right]=\prod_{i=1}^{n}\frac{(\phi+\mu_i)(-\hat{\lambda}_i(1))}{\mu_i(\phi-\hat{\lambda}_i(1))},\quad Re(\phi)\geq 0. \tag{3.53}$$

This result is in accordance with a result in [24].

The virtual waiting time

By inserting (3.17) into (3.49), we have for this system

$$\begin{aligned}F(\phi,\eta,\gamma)=&\frac{B_1(\phi)}{\prod_{i=1}^n(\phi-\lambda_i(1,\eta))}\left[\sum_{i=1}^{k}\sum_{j=1}^{\gamma}\sum_{l=0}^{j-1}q_{ij}\frac{h_i^{(l)}(1,\eta)}{(\phi+w_i)^{j-l}}\right]\\&+\frac{B_1(\phi)}{\prod_{i=1}^n(\phi-\lambda_i(1,\eta))}C^{-}(1,0,\eta,\gamma)\\&-\prod_{i=1}^{n}\frac{(\phi+\mu_i)}{(\phi-\lambda_i(1,\eta))}\left[\sum_{i=1}^{k}\sum_{j=1}^{\gamma}\sum_{l=0}^{j-1}q_{ij}\frac{h_i^{(l)}(1,\eta)}{(\phi+w_i)^{j-l}}\right]\\&-\prod_{i=1}^{n}\frac{(\phi+\mu_i)}{(\phi-\lambda_i(1,\eta))}C^{-}(1,0,\eta,\gamma)+\frac{P(\phi)^\gamma}{\prod_{i=1}^k(\phi+w_i)^\gamma}.\end{aligned} \tag{3.54}$$

Then, by inserting (3.54) and (3.18) into (3.20) we obtain for $(Re(\phi)\geq 0, Re(\eta)>0,\gamma\geq 0)$ or $(Re(\phi)>0, Re(\eta)\geq 0,\gamma\geq 0)$,

$$\begin{aligned}Z^*(\phi,\eta,\gamma)=&\frac{B_1(\phi)}{(\eta-\phi)\prod_{i=1}^n(\phi-\lambda_i(1,\eta))}\left[\sum_{i=1}^{k}\sum_{j=1}^{\gamma}\sum_{l=0}^{j-1}q_{ij}\frac{h_i^{(l)}(1,\eta)}{(\phi+w_i)^{j-l}}\right]\\&+\frac{B_1(\phi)}{(\eta-\phi)\prod_{i=1}^n(\phi-\lambda_i(1,\eta))}C^{-}(1,0,\eta,\gamma)\\&-\frac{1}{(\eta-\phi)}\prod_{i=1}^{n}\frac{(\phi+\mu_i)}{(\phi-\lambda_i(1,\eta))}\left[\sum_{i=1}^{k}\sum_{j=1}^{\gamma}\sum_{l=0}^{j-1}q_{ij}\frac{h_i^{(l)}(1,\eta)}{(\phi+w_i)^{j-l}}\right]\\&-\frac{1}{(\eta-\phi)}\prod_{i=1}^{n}\frac{(\phi+\mu_i)}{(\phi-\lambda_i(1,\eta))}C^{-}(1,0,\eta,\gamma)\\&+\frac{P(\phi)^\gamma}{(\eta-\phi)\prod_{i=1}^k(\phi+w_i)^\gamma}-\frac{F(\eta,\eta,\gamma)}{(\eta-\phi)}+\frac{F(\eta,\eta,\gamma)}{\eta}.\end{aligned} \tag{3.55}$$

Again, we get a rational function in ϕ that allows us to invert it with respect to this variable analytically to find an explicit expression for the Laplace transform

$$\tilde{z}(x,\eta,\gamma) = \int_0^\infty e^{-\eta t} P(V_t \le x | C_0 = \gamma) dt, \quad x \ge 0, Re(\eta) \ge 0, \gamma \ge 0. \tag{3.56}$$

By defining

$$s_i(\eta) = \frac{B_1(\lambda_i(1,\eta))}{\lambda_i(1,\eta)(\eta - \lambda_i(1,\eta)) \prod_{j=1,j\neq i}^n (\lambda_i(1,\eta) - \lambda_j(1,\eta))}$$

and

$$t_i(\eta) = \frac{\prod_{j=1}^n (\lambda_i(1,\eta) + \mu_j)}{\lambda_i(1,\eta)(\eta - \lambda_i(1,\eta)) \prod_{j=1,j\neq i}^n (\lambda_i(1,\eta) - \lambda_j(1,\eta))},$$

we have for $x \ge 0, Re(\eta) \ge 0, \gamma \ge 0$,

$$\begin{aligned} \tilde{z}(x,\eta,\gamma) = &\frac{1}{\eta} + \sum_{i=1}^n (s_i(\eta) - t_i(\eta))(C^+(1,\lambda_i(1,\eta,\gamma),\eta) + C^-(1,0,\eta,\gamma))e^{\lambda_i(1,\eta)x} \\ &+ \sum_{i=1}^k \sum_{j=1}^\gamma \sum_{l=0}^{j-1} q_{ij} h_i^{(l)}(1,\eta) \sum_{m=1}^{j-l} \frac{\Phi_{1m}(-w_i,\eta) x^{j-l-m} e^{-w_i x}}{(j-l-m)!(m-1)!} \\ &- \sum_{i=1}^k \sum_{j=1}^\gamma \sum_{l=0}^{j-1} q_{ij} h_i^{(l)}(1,\eta) \sum_{m=1}^{j-l} \frac{\Phi_{2m}(-w_i,\eta) x^{j-l-m} e^{-w_i x}}{(j-l-m)!(m-1)!} \\ &+ \sum_{i=1}^k \sum_{j=1}^\gamma \frac{\Phi_{3ij}(-w_i,\eta,\gamma) x^{\gamma-j} e^{-w_i x}}{(\gamma-j)!(j-1)!}, \end{aligned} \tag{3.57}$$

where

$$\begin{aligned} \Phi_{1m}(\phi,\eta) &= \frac{\partial^{m-1}}{\partial \phi^{m-1}} \left[\frac{B_1(\phi)}{\phi(\eta-\phi)\prod_{m=1}^n (\phi - \lambda_m(1,\eta))} \right], \\ \Phi_{2m}(\phi,\eta) &= \frac{\partial^{m-1}}{\partial \phi^{m-1}} \left[\frac{1}{\phi(\eta-\phi)} \prod_{i=1}^n \frac{(\phi+\mu_i)}{(\phi - \lambda_i(1,\eta))} \right], \\ \Phi_{3ij}(\phi,\eta,\gamma) &= \frac{\partial^{lj-1}}{\partial \phi^{j-1}} \left[\frac{P(\phi)^\gamma}{\phi(\eta-\phi)\prod_{m=1,m\neq i}^n (\phi + w_m)^\gamma} \right]. \end{aligned}$$

The Laplace-Stieltjes transform of the probability distribution function of the virtual waiting time in steady state is easily found from (3.23) by substituting the expression for $Z(\phi)$ in (3.53) obtaining for $Re(\phi) \ge 0$,

$$\begin{aligned} Z^*(\phi) &= 1 - \rho + Z(\phi) \frac{1 - B(\phi)}{\alpha \phi} \\ &= 1 - \rho + \prod_{i=1}^n \frac{(-\hat{\lambda}_i(1))}{(\mu_i)(\phi - \hat{\lambda}_i(1))} \frac{(\prod_{j=1}^n (\phi + \mu_j) - B_1(\phi))}{\alpha \phi}, \end{aligned} \tag{3.58}$$

as given in de Smit[24]. By inverting the Laplace-Stieltjes transform (3.58) we obtain the probability distribution function of the virtual waiting time in steady state. As a result, we have for $x \geq 0$,

$$\begin{aligned} P(V \leq x) =& 1 - \rho + \frac{1}{\alpha} \prod_{j=1}^{n} \frac{(-\hat{\lambda}_j(1))}{\mu_j} \sum_{i=1}^{n} \frac{(\prod_{j=1}^{n} (\hat{\lambda}_i(1) + \mu_j) - B_1(\hat{\lambda}_i(1))) e^{\hat{\lambda}_i(1) x}}{\hat{\lambda}_i(1)^2 \prod_{j=1, j \neq i} (\hat{\lambda}_i(1) - \hat{\lambda}_j(1))} \\ &+ \frac{1}{\alpha} \left(\sum_{i=1}^{n} \frac{1}{\mu_i} - \frac{B_1'(0)}{\prod_{i=1}^{n} \mu_i} \right). \end{aligned} \tag{3.59}$$

Number of customers at arrival epochs

For the queueing system under consideration, the integral in the third term of (3.31) would be

$$\begin{aligned} &\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} (1 - rA(-\xi))^{-1} A(-\xi) B(\xi) Z(r, \xi, 0, \gamma) \\ &= \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \frac{A(-\xi) B_1(\xi) \left[C^+(r, \xi, 0, \gamma) + C^-(r, 0, 0, \gamma)\right]}{(1 - rA(-\xi)) \prod_{i=1}^{n} (\xi - \lambda_i(r, 0))} \\ &= 2\pi i \sum_{i=1}^{n} \frac{A(-\lambda_i(r,0)) B_1(\lambda_i(r,0)) \left[C^+(r, \lambda_i(r,0), 0, \gamma) + C^-(r, 0, 0, \gamma)\right]}{\lambda_i(r,0)(1 - rA(-\lambda_i(r,0))) \prod_{j=1, j\neq i}^{n} (\lambda_i(r,0) - \lambda_j(r,0))} \\ &\quad + 2\pi i \sum_{i=1}^{k} \frac{\partial^\gamma}{\partial \xi^\gamma} \left(\frac{A(-\xi) B_1(\xi) p_i \left(K^-(1, -w_i, 0)\right)^{-1}}{\xi (1 - rA(-\xi)) \prod_{j=1}^{n} (\xi - \lambda_j(1, 0))} \right) \Bigg|_{\boldsymbol{\xi = -w_i}}. \end{aligned} \tag{3.60}$$

If we substitute this into (3.31) we obtain for $|r| \leq 1, \gamma \geq 0$,

$$\begin{aligned} U(r, \gamma) =& \gamma + \frac{r}{(1-r)^2} \\ &- r \sum_{j=1}^{\gamma} \frac{1}{(j-1)!} \sum_{i=1}^{k} \frac{d^j}{d\xi^j} \left[\frac{A(-\xi) P(\xi)^j}{\xi (1 - rA(-\xi)) \prod_{n=1, n \neq i}^{k} (\xi + w_n)^j} \right]_{\xi = -w_i} \\ &- r \sum_{i=1}^{n} \frac{A(-\lambda_i(r,0)) B_1(\lambda_i(r,0)) \left[C^+(r, \lambda_i(r,0), 0, \gamma) + C^-(r, 0, 0, \gamma)\right]}{\lambda_i(r,0)(1 - rA(-\lambda_i(r,0))) \prod_{j=1, j\neq i}^{n} (\lambda_i(r,0) - \lambda_j(r,0))} \\ &- \sum_{i=1}^{k} \frac{\partial^\gamma}{\partial \xi^\gamma} \left(\frac{A(-\xi) B_1(\xi) p_i \left(K^-(1, -w_i, 0)\right)^{-1}}{\xi (1 - rA(-\xi)) \prod_{j=1}^{n} (\xi - \lambda_j(1, 0))} \right) \Bigg|_{\boldsymbol{\xi = -w_i}}, \end{aligned} \tag{3.61}$$

and if we invert it we get $E[C_n], n = 0, 1, \cdots$. Meanwhile, from (3.32), the expectation of steady-state number of customers at arrival epochs is

$$\begin{aligned} E[C] = -1 &- \sum_{i=1}^{n} \frac{A(-\hat{\lambda}_i(1)) B_1(\hat{\lambda}_i(1)) \left[C^+(1, \hat{\lambda}_i(1), 0, \gamma) + C^-(1, 0, 0, \gamma)\right]}{\hat{\lambda}_i(1)(1 - A(-\hat{\lambda}_i(1))) \prod_{j=1, j\neq i}^{n} (\hat{\lambda}_i(1) - \hat{\lambda}_j(1))} \\ &- \sum_{i=1}^{k} \frac{\partial^\gamma}{\partial \xi^\gamma} \left(\frac{B_1(\xi) p_i K^-(1, -w_i, 0)^{-1}}{\xi (1 - A(-\xi)) \prod_{j=1}^{n} (\xi - \lambda_j(1, 0))} \right) \Bigg|_{\boldsymbol{\xi = -w_i}}. \end{aligned} \tag{3.62}$$

Number of customers in continuous time

To get the explicit expression for the expected number of customers in continuous time in steady-state, we consider the following.

$$\begin{aligned}
\lim_{\eta\downarrow 0}\eta Z(1,\phi,\eta,\gamma) &= \lim_{\eta\downarrow 0}\eta\left[\prod_{i=1}^{n}\frac{(\phi+\mu_i)}{(\phi-\lambda_i(1,\eta))}\sum_{j=1}^{k}p_j\frac{[K^-]^{-1}(1,-w_j,\eta)}{(\phi+w_j)^\gamma}\right.\\
&\qquad\left.+\prod_{i=1}^{n}\frac{(\phi+\mu_i)}{(\phi-\lambda_i(1,\eta))}C^-(r,0,\eta,\gamma)\right]\\
&= \lim_{\eta\downarrow 0}\eta\prod_{i=1}^{n}\frac{(\phi+\mu_i)}{(\phi-\lambda_i(1,\eta))}\frac{\prod_{i=1}^{n}-\lambda_i(1,\eta)}{\prod_{i=1}^{n}\mu_i(1-A(\eta))}\\
&= \prod_{i=1}^{n}\frac{(\phi+\mu_i)(-\lambda_i(1,0))}{\mu_i(-A'(0))(\phi-\lambda_i(1,0))}\\
&= \frac{1}{\alpha}\prod_{i=1}^{n}\frac{(\phi+\mu_i)(-\lambda_i(1,0))}{\mu_i(\phi-\lambda_i(1,0))}.
\end{aligned} \tag{3.63}$$

If we substitute this into (3.40) we obtain

$$\begin{aligned}
&E[C^*]\\
&= -\frac{1}{2\pi i\alpha}\int_{-i\infty-0}^{i\infty-0}\frac{d\xi}{\xi^2}B_1(\xi)\prod_{i=1}^{n}\frac{-\lambda_i(1,0)}{(\xi-\lambda_i(1,0))\mu_i}\\
&= -\prod_{i=1}^{n}\frac{(-\hat{\lambda}_i(1))}{\alpha\mu_i}\left[\frac{B_1'(0)\prod_{i=1}^{n}(-\hat{\lambda}_i(1))-\prod_{i=1}^{n}\mu_i\sum_{i=1}^{n}\prod_{j=1,j\neq i}^{n}(-\hat{\lambda}_j(1))}{\prod_{i=1}^{n}(-\hat{\lambda}_i(1))^2}\right].
\end{aligned} \tag{3.64}$$

3.7.2 The system $K_m/G/1$

The Laplace-Stieltjes transform of the inter-arrival times for this model has the form

$$A(\phi) = \frac{A_1(\phi)}{\prod_{i=1}^{m}(\phi+\lambda_i)},$$

where $Re(\lambda_i) > 0$, $i = 1, 2, \ldots, m$, and $A_1(\phi)$ is a polynomial of degree $(m-1)$ or less. Now we have

$$1 - rA(\eta-\phi)B(\phi) = \frac{\prod_{i=1}^{m}(\eta-\phi+\lambda_i) - rA_1(\eta-\phi)B(\phi)}{\prod_{i=1}^{m}(\eta-\phi+\lambda_i)}. \tag{3.65}$$

With a similar proof as for the case $GI/K_n/1$, it can be shown that for $(|r| < 1, Re(\eta) \geq 0)$ or $(|r| = 1, \rho < 1, Re(\eta) \geq 0)$, the numerator of (3.65) has exactly m zeros in the right half-plane $Re(\phi) > 0$, which we denote by $\mu_1(r,\eta), \mu_2(r,\eta), \ldots, \mu_m(r,\eta)$, so that

$$1 - rA(\eta-\phi)B(\phi) = K^+(r,\phi,\eta)K^-(r,\phi,\eta), \quad Re(\phi) = 0,$$

with

$$K^+(r,\phi,\eta) = \frac{\prod_{i=1}^m (\eta-\phi+\lambda_i) - rA_1(\eta-\phi)B(\phi)}{\prod_{i=1}^m(\phi-\mu_i(r,\eta))},$$

and

$$K^-(r,\phi,\eta) = \prod_{i=1}^m \frac{(\phi-\mu_i(r,\eta))}{(\eta-\phi+\lambda_i)}.$$

It is clear that $K^+(r,\phi,\eta)$ satisfies $\tilde{A}^+$ and $K^-(r,\phi,\eta)$ satisfies $\tilde{A}^-$. We will need the partial fractions expansion

$$\begin{aligned}\frac{P(\phi)^\gamma}{\prod_{i=1}^k(\phi+w_i)^\gamma}\left[K^-(r,\phi,\eta)\right]^{-1} &= \frac{P(\phi)^\gamma}{\prod_{i=1}^k(\phi+w_i)^\gamma}\prod_{i=1}^m\frac{(\eta-\phi+\lambda_i)}{(\phi-\mu_i(r,\eta))}\\ &= P(\phi)^\gamma\frac{\prod_{j=1}^m(\eta-\phi+\lambda_j)}{\prod_{i=1}^{k+m}(\phi-a_i(r,\eta))^{l_i}}\\ &= \sum_{i=1}^{k+m}\sum_{j=1}^{l_i}\frac{e_{ij}(r,\eta,\gamma)}{(\phi-a_i(r,\eta))^{l_i-j+1}},\end{aligned} \tag{3.66}$$

where

$$a_i(r,\eta) = \begin{cases} -w_i & \text{for} \quad i=1,\dots k,\\ \mu_{i-k}(r,\eta) & \text{for} \quad i=k+1,\dots,k+m,\end{cases}$$

$$l_i = \begin{cases} \gamma & \text{for} \quad i=1,\dots k,\\ 1 & \text{for} \quad i=k+1,\dots,k+m.\end{cases}$$

If we denote by

$$R(\phi,\eta,\gamma) = P(\phi)^\gamma\prod_{j=1}^m(\eta-\phi+\lambda_j)$$

and

$$Q(\phi,\eta) = \prod_{i=1}^{k+m}(\phi-a_i(r,\eta))^{l_i},$$

then the functions $e_{ij}(r,\eta)$ satisfy

$$e_{ij}(r,\eta,\gamma) = \frac{1}{(j-1)!}\frac{\partial^{j-1}}{\partial\phi^{j-1}}(\phi-a_i(r,\eta))^{l_i}\frac{R(\phi,\eta,\gamma)}{Q(\phi,\eta)}\Bigg|_{\boldsymbol{\phi=a_i(r,\eta)}}. \tag{3.67}$$

We now choose the following decomposition in (3.7)

$$C^+(r,\phi,\eta,\gamma) = \sum_{i=1}^k\sum_{j=1}^\gamma\frac{e_{ij}(r,\eta,\gamma)}{(\phi+w_i)^{\gamma-j+1}},$$

and

$$C^-(r,\phi,\eta,\gamma) = \sum_{i=k+1}^{k+m}\frac{e_{ij}(r,\eta,\gamma)}{\phi-\mu_{i-k}(r,\eta)}.$$

It is clear that $C^+(r,\phi,\eta,\gamma)$ satisfies A^+ and $C^-(r,\phi,\eta,\gamma)$ satisfies A^-.

The actual waiting time

From (3.8) we have for $(|r| < 1, Re(\phi) \geq 0, Re(\eta) \geq 0, \gamma \geq 0)$, or $(|r| \leq 1, Re(\phi) \geq 0, Re(\eta) > 0, \gamma \geq 0)$, or $(|r| \leq 1, Re(\phi) > 0, Re(\eta) \geq 0, \gamma \geq 0)$,

$$\begin{aligned} Z(r,\phi,\eta,\gamma) = & \left[\sum_{i=1}^{k}\sum_{j=1}^{\gamma}\frac{e_{ij}(r,\eta)}{(\phi+w_i)^{\gamma}} + C^{-}(r,0,\eta,\gamma)\right] \\ & \cdot\left[\frac{\prod_{i=1}^{m}(\phi-\mu_i(r,\eta))}{\prod_{i=1}^{m}(\eta-\phi+\lambda_i) - rA_1(\eta-\phi)B(\phi)}\right]. \end{aligned} \tag{3.68}$$

This function, in general, is not rational. Hence, we can not derive an explicit expression for the generating function $G(r,x,\gamma)$ from (3.68). We just can get the generating functions of the kth moments of W_n by differentiating the function $Z(r,\phi,0,\gamma)$ with respect to the variable ϕ.

For $\rho < 1$, we consider the steady-state distribution of the waiting times. For this purpose, we need to study the behavior of the zeros $\mu_i(r,\eta)$ for $r \uparrow 1$ and $\eta = 0$. Take $\delta > 0$ and consider the contour $C^{+}_{\delta,R}$. Then for R large enough $|A(-\phi)B(\phi)| < 1$ on the semi-circle $|\phi| = R, Re(\phi) < 0$, whereas on $Re(\phi) = -\delta$,

$$\begin{aligned} |A(-\phi)B(\phi)| \leq |A(\delta)B(-\delta)| &= (1-\alpha\delta+o(\delta))(1+\beta\delta+o(\delta)) \\ &= 1-\alpha\delta(1-\rho)+o(\delta), \delta\downarrow 0. \end{aligned}$$

Hence, $|A(-\phi)B(\phi)| < 1$ on $Re(\phi) = -\delta$ if $o(\delta)/\delta < \alpha(1-\rho)$, which will hold for $\rho < 1$ and δ small enough. With the aid of Rouché's theorem, we now see that for $\rho < 1$, $1-A(-\phi)B(\phi)$ has a simple zero at the origin, which we denote by $\mu_1(1,0)$, and has $m-1$ zeros in the right half-plane $Re(\phi) \geq 0$, which we denote by $\mu_2(1,0),\ldots,\mu_m(1,0)$. Since the $\mu_i(r,0)$ are continuous functions in r for $|r| \leq 1$, we may write $\mu_i(1,0) = \lim_{r\uparrow 1}\mu_i(r,0)$, $i = 1,\ldots,m$.

By definition, we see that

$$\lim_{r\uparrow 1}(1-r)C^{+}(r,\phi,0,\gamma)K^{+}(r,\phi,0)^{-1} = 0.$$

Moreover,

$$\begin{aligned} &\lim_{r\uparrow 1}(1-r)C^{-}(r,0,0,\gamma)K^{+}(r,\phi,0)^{-1} \\ &= \lim_{r\uparrow 1}\frac{(1-r)}{-a_{k+1}(r,0)}e_{(k+1)1}(r,0,\gamma)K^{+}(r,\phi,0)^{-1} \\ &= \lim_{r\uparrow 1}\frac{(1-r)}{-\mu_1(r,0)}e_{(k+1)1}(r,0,\gamma)K^{+}(r,\phi,0)^{-1}. \end{aligned} \tag{3.69}$$

To determine this limit, we use the fact $1-rA(-\mu_1(r,0))B(\mu_1(r,0)) = 0$, from which it is readily verified that

$$\mu_1'(1,0) = 1/(\beta-\alpha). \tag{3.70}$$

Moreover, from the definition of $e_{ij}(r, \eta, \gamma)$ in (3.67) we see that

$$\lim_{r \uparrow 1} e_{(k+1)1}(r, 0, \gamma) = \frac{\prod_{i=1}^{m} \lambda_i}{\prod_{i=2}^{m}(-\mu_i(1,0))},$$

which shows that this limit is independent of γ. By combining this result with (3.70) we obtain

$$\lim_{r \uparrow 1}(1-r)C^-(r,0,0,\gamma)K^+(r,\phi,0)^{-1} = (\beta-\alpha)\frac{\prod_{i=1}^{m} \lambda_i}{\prod_{i=2}^{m}(-\mu_i(1,0))}K^+(1,\phi,0)^{-1}.$$

Then from (3.11) we have for $Re(\phi) \geq 0$,

$$Z(\phi) = \phi(\beta-\alpha)\left(\prod_{i=2}^{m}\frac{\mu_i(1,0)-\phi}{\mu_i(1,0)}\right)\frac{\prod_{i=1}^{m}\lambda_i}{\prod_{i=1}^{m}(\lambda_i-\phi)-A_1(-\phi)B(\phi)} \tag{3.71}$$

in accordance with a result in de Smit[24].

The virtual waiting time

For this system we have for $Re(\phi) \geq 0, Re(\eta) \geq 0, \gamma \geq 0$,

$$\begin{aligned} F(\phi,\eta,\gamma) = &\left[\sum_{i=1}^{k}\sum_{j=1}^{\gamma}\frac{e_{ij}(1,\eta,\gamma)}{(\phi+w_i)^\gamma}+C^-(1,0,\eta,\gamma)\right] \\ &\cdot\left[\frac{\prod_{i=1}^{m}(\phi-\mu_i(1,\eta))}{\prod_{i=1}^{m}(\eta-\phi+\lambda_i)-A_1(\eta-\phi)B(\phi)}\right]B(\phi) \\ &-\left[\sum_{i=1}^{k}\sum_{j=1}^{\gamma}\frac{e_{ij}(1,\eta)}{(\phi+w_i)^\gamma}+C^-(1,0,\eta,\gamma)\right] \\ &\cdot\left[\frac{\prod_{i=1}^{m}(\phi-\mu_i(1,\eta))}{\prod_{i=1}^{m}(\eta-\phi+\lambda_i)-A_1(\eta-\phi)B(\phi)}\right]+\frac{P(\phi)^\gamma}{\prod_{i=1}^{k}(\phi+w_i)^\gamma}, \end{aligned} \tag{3.72}$$

so that for $Re(\phi) \geq 0, Re(\eta) \geq 0, \gamma \geq 0$,

$$\begin{aligned} Z^*(\phi,\eta,\gamma) = &\left[\sum_{i=1}^{k}\sum_{j=1}^{\gamma}\frac{e_{ij}(1,\eta,\gamma)}{(\phi+w_i)^\gamma}+C^-(1,0,\eta,\gamma)\right] \\ &\cdot\left[\frac{\prod_{i=1}^{m}(\phi-\mu_i(r))}{\prod_{i=1}^{m}(\eta-\phi+\lambda_i)-A_1(\eta-\phi)B(\phi)}\right]\frac{B(\phi)}{\eta-\phi} \\ &-\left[\sum_{i=1}^{k}\sum_{j=1}^{\gamma}\frac{e_{ij}(1,\eta,\gamma)}{(\phi+w_i)^\gamma}+C^-(1,0,\eta,\gamma)\right] \\ &\cdot\left[\frac{\prod_{i=1}^{m}(\phi-\mu_i(1,\eta))}{\prod_{i=1}^{m}(\eta-\phi+\lambda_i)-A_1(\eta-\phi)B(\phi)}\right]\frac{1}{(\eta-\phi)} \\ &+\frac{P(\phi)^\gamma}{(\eta-\phi)\prod_{i=1}^{k}(\phi+w_i)^\gamma}-\frac{F(\eta,\eta,\gamma)}{(\eta-\phi)}+\frac{F(\eta,\eta,\gamma)}{\eta}. \end{aligned} \tag{3.73}$$

The time-dependent distribution of the virtual waiting time can be obtained by inverting this transfom.

The Laplace-Stieltjes transform of the steady-state probability distribution of the virtual waiting time follows from (3.23) and (3.71) and is given by

$$\begin{aligned} Z^*(\phi) &= 1-\rho+Z(\phi)\frac{1-B(\phi)}{\alpha\phi} \\ &= 1-\rho+\frac{(\beta-\alpha)}{\alpha}\left(\prod_{i=2}^{m}\frac{\mu_i(1,0)-\phi}{\mu_i(1,0)}\right)\frac{\prod_{i=1}^{m}\lambda_i(1-B(\phi))}{\prod_{i=1}^{m}(\lambda_i-\phi)-A_1(-\phi)B(\phi)}, \end{aligned} \tag{3.74}$$

for $Re(\phi)\geq 0, \rho<1$. This result is an accordance with a result in de Smit[24].

The number of customers at arrival epochs

The last integral on the right hand side of (3.31) for the system under consideration becomes

$$\begin{aligned} &\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}(1-rA(-\xi))^{-1}A(-\xi)B(\xi)Z(r,\xi,0,\gamma) \\ &=\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}\frac{B(\xi)Z(1,\xi,0,\gamma)\prod_{i=1}^{m}(-\xi+\lambda_i)}{\prod_{i=1}^{m}(-\xi+\lambda_i)-rA_1(-\xi)}. \end{aligned} \tag{3.75}$$

We have discussed on page 38 that $\mu_i(r,0), i=1,2,\cdots,m$, are the zeroes of the denominator of (3.75). Then (3.75) becomes

$$2\pi i\sum_{i=1}^{m}\frac{B(\mu_i(r,0))Z(1,\mu_i(r,0),0,\gamma)\prod_{j=1}^{m}(-\mu_i(r,0)+\lambda_j)}{\prod_{j=1,j\neq i}^{m}(\mu_i(r,0)-\mu_j(r,0))}.$$

If we substitute this into (3.31), then we obtain the explicit expression for the generating function of the expectations $E[C_n|C_0=\gamma], n=0,1,\ldots,$ which for $|r|<1, \gamma\geq 0$, is given by

$$\begin{aligned} U(r,\gamma)=&\gamma+\frac{r}{(1-r)^2} \\ &-r\sum_{j=1}^{\gamma}\frac{1}{(j-1)!}\sum_{i=1}^{k}\frac{d^j}{d\xi^j}\left[\frac{A(-\xi)P(\xi)^j}{\xi(1-rA(-\xi))\prod_{n=1,n\neq i}^{k}(\xi+w_n)^j}\right]_{\xi=-w_i} \\ &-r\sum_{i=1}^{m}\frac{B(\mu_i(r,0))Z(1,\mu_i(r,0),0,\gamma)\prod_{j=1}^{m}(-\mu_i(r,0)+\lambda_j)}{\prod_{j=1,j\neq i}^{m}(\mu_i(r,0)-\mu_j(r,0))}. \end{aligned} \tag{3.76}$$

The number of customers in continuous time

For the system under consideration we have the following relations.

$$\begin{aligned}
&\lim_{\eta\downarrow 0} \eta Z(1,\phi,\eta,\gamma) \\
&= \lim_{\eta\downarrow 0} \eta \left[\sum_{i=1}^{k}\sum_{j=1}^{\gamma} \frac{e_{ij}(1,\eta,\gamma)}{(\phi+w_i)^{\gamma}} + C^{-}(1,0,\eta,\gamma)\right] \\
&\quad \cdot \left[\frac{\prod_{i=1}^{m}(\phi-\mu_i(1,\eta))}{\prod_{i=1}^{m}(\eta-\phi+\lambda_i) - A_1(\eta-\phi)B(\phi)}\right] \\
&= 0 + \lim_{\eta\downarrow 0} \eta C^{-}(1,0,\eta,\gamma) \frac{\prod_{i=1}^{m}(\phi-\mu_i(1,\eta))}{\prod_{i=1}^{m}(\eta-\phi+\lambda_i) - A_1(\eta-\phi)B(\phi)} \\
&= \lim_{\eta\downarrow 0} \left(\eta \sum_{i=k+1}^{k+m} \frac{e_{i1}(1,\eta,\gamma)}{-\mu_{i-k}(1,\eta)}\right) \frac{\prod_{i=1}^{m}(\phi-\mu_i(1,\eta))}{\prod_{i=1}^{m}(\eta-\phi+\lambda_i) - A_1(\eta-\phi)B(\phi)} \\
&= \lim_{\eta\downarrow 0} \frac{\eta e_{(k+1)1}(1,\eta,\gamma)}{-\mu_1(1,\eta)} \frac{\prod_{i=1}^{m}(\phi-\mu_i(1,\eta))}{\prod_{i=1}^{m}(\eta-\phi+\lambda_i) - A_1(\eta-\phi)B(\phi)}.
\end{aligned} \tag{3.77}$$

To determine this limit, we need to differentiate the equation

$$1 - A(\eta-\mu_1(1,\eta))B(\mu_1(1,\eta)) = 0.$$

It is readily verified that

$$\lim_{\eta\downarrow 0} \mu_1'(1,\eta) = \frac{\alpha}{\alpha-\beta}. \tag{3.78}$$

By using l'Hôpital's rule we obtain

$$\lim_{\eta\downarrow 0} \eta Z(1,\phi,\eta,\gamma) = \frac{1}{\alpha} Z(\phi). \tag{3.79}$$

If we substitute (3.79) into (3.40), we obtain

$$\begin{aligned}
&E[C^*] \\
&= \frac{(\alpha-\beta)}{2\pi i\alpha} \frac{\prod_{i=1}^{m}\lambda_i}{\prod_{i=2}^{m}\mu_i(1,0)} \int_{-i\infty-0}^{i\infty-0} \frac{d\xi}{\xi} B(\xi) \frac{\prod_{i=2}^{m}(\mu_i(1,0)-\xi)}{\prod_{i=1}^{m}(\lambda_i-\xi) - A_1(-\xi)B(\xi)}.
\end{aligned} \tag{3.80}$$

It is clear that $\xi = 0$ is a pole of the integrand of order 2. Then for $\rho < 1$ and since the probability distribution of A_n is non-lattice,

$$E[C^*] = \frac{c(\beta-\alpha)}{\alpha} \frac{\prod_{i=1}^{m}\lambda_i}{\prod_{i=2}^{m}\mu_i(1,0)}, \tag{3.81}$$

where c is the residue of

$$\frac{B(\xi)}{\xi} \frac{\prod_{i=2}^{m}(\mu_i(1,0)-\xi)}{\prod_{i=1}^{m}(\lambda_i-\xi) - A_1(-\xi)B(\xi)}$$

at $\xi = 0$.

3.7.3 Examples

In this section we give some examples of time-dependent distributions of the waiting times and the number of customers. To get all distribution functions and expectations of interest, we apply the numerical inversion algorithms proposed in [3] to the related transforms.

We suppose in all examples that the number of special customers in the system is $C_0 = \gamma$. The service time of these special customers is exponentially distributed with mean $\frac{2}{3}$. So that

$$E\left[e^{-\phi W_0}\right] = \left(\frac{1.5}{\phi + 1.5}\right)^{\gamma}.$$

1. *The system $H_2/M/1$*
   We suppose that the inter-arrival times have a H_2 distribution with Laplace-Stieltjes transform

   $$A(\phi) = \frac{(15 + 4\phi)}{(3 + \phi)(5 + \phi)},$$

   and the service times have an exponential distribution with the Laplace-Stieltjes transform

   $$B(\phi) = \frac{6}{\phi + 6}.$$

   Note that the traffic intensity $\rho = \frac{15}{24}$. This system is a special case of the $GI/K_n/1$ system, so we will follow the analysis in sub-section 3.7.1 in order to obtain the distributions of interest.

   The Wiener-Hopf type equation (3.41) for this system is

   $$1 - rA(\eta - \phi)B(\phi) = \frac{(3 + \eta - \phi)(5 + \eta - \phi)(\phi + 6) - 6r(15 + 4(\eta - \phi))}{(3 + \eta - \phi)(5 + \eta - \phi)(\phi + 6)}. \quad (3.82)$$

   For $\eta = 0$ and $r \uparrow 1$ the numerator of (3.82) has exactly one zero in the left half-plane $Re(\phi) < 0$, that is $\lambda(1,0) = -2.162$.

   The steady-state distribution function of the actual waiting time, from (3.53), is

   $$P(W \leq x) = -\frac{\lambda(1,0)}{6} + \frac{\lambda(1,0) + 6}{6}\left(1 - e^{\lambda(1,0)x}\right).$$

   After following the analysis in sub-section 3.7.1, the distribution function $P(W_n \leq x)$ for fixed n can be obtained by inverting the generating function (3.52) numerically, where we apply the numerical algorithm proposed in [3]. The behavior of W_n for some values of n is shown in figure 3.1.

   The steady-state distribution function of the virtual waiting time follows from (3.58),

   $$P(V \leq x) = 1 - \rho + \rho\left(1 - e^{\lambda(1,0)x}\right).$$

   The distribution function $P(V_t \leq x)$ for fixed t is obtained by inverting the Laplace transform (3.57) numerically. The behavior of the distribution function of V_t for some values of t is shown in figure 3.2.

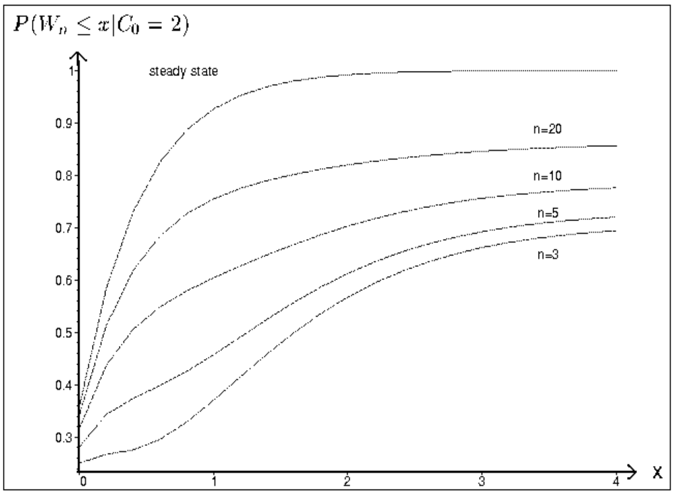

Figure 3.1: $P(W_n \leq x|C_0 = 2)$ for some n.

We invert (3.36) numerically to get the expectation of the number of customers in continuous time. Note from (3.64) that

$$E\left[C^*\right] = -\frac{1}{\alpha\lambda(1,0)} = 1.7342.$$

The behavior of $E[C_t^*]$ as t increases is shown in figure 3.3.

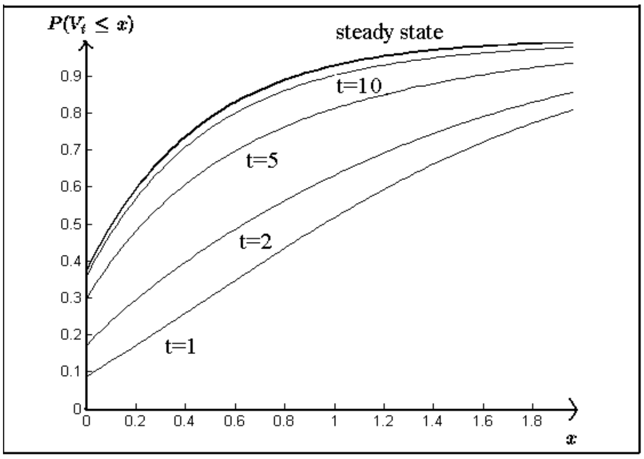

Figure 3.2: $P(V_t \leq x|C_0 = 2)$ for some t.

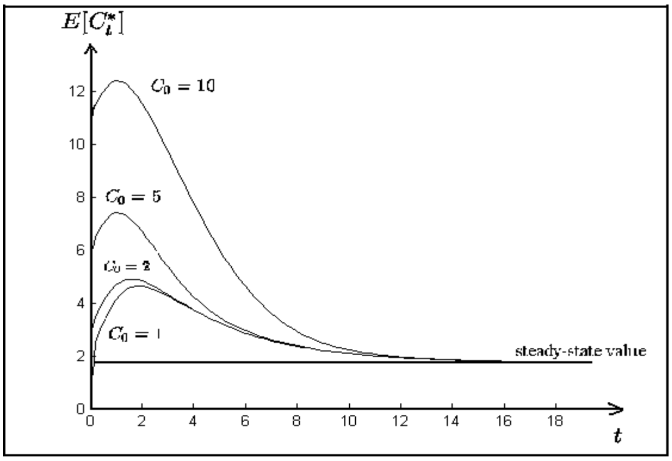

Figure 3.3: $E[C^*_t|C_0 = \gamma]$ as t increases, for some values of γ.

2. *The system $H_2/D/1$*
   In this example we suppose that the inter-arrival times have a H_2 distribution with the Laplace-Stieltjes transform

$$A(\phi) = \frac{(15+4\phi)}{(3+\phi)(5+\phi)},$$

the service times have a deterministic distribution with the Laplace-Stieltjes transform

$$B(\phi) = e^{-\beta\phi}.$$

Note that the traffic intensity $\rho = \frac{15\beta}{4}$. This system is a special case of $K_m/G/1$ system, so we will follow the analysis in sub-section 3.7.2 in order to obtain the distributions of interest.

The Wiener-Hopf type equation (3.65) for this system is

$$1 - rA(\eta-\phi)B(\phi) = \frac{(3+\eta-\phi)(5+\eta-\phi) - r(15+4(\eta-\phi))e^{-\beta\phi}}{(3+\eta-\phi)(5+\eta-\phi)}. \quad (3.83)$$

For fixed η with $Re(\eta) > 0$ the numerator of (3.83) has two zeros in the right half-plane $Re(\phi) > 0$. For this system, we obtain the factors

$$K^+(r,\phi,\eta) = \frac{(3+\eta-\phi)(5+\eta-\phi) - r(15+4(\eta-\phi))e^{-\beta\phi}}{(\phi-\mu_1(r,\eta))(\phi-\mu_2(r,\eta))},$$
$$K^-(r,\phi,\eta) = \frac{(\phi-\mu_1(r,\eta))(\phi-\mu_2(r,\eta))}{(3+\eta-\phi)(5+\eta-\phi)}$$

For the decomposition in (3.7), we derive a partial fraction expansion of $E[e^{-\phi W_0}]K^-(1,\phi,\eta)^{-1}$, i.e.

$$\begin{aligned} &E[e^{-\phi W_0}]K^-(1,\phi,\eta)^{-1} \\ =&\frac{A_1(\eta)}{(\phi+1.5)} + \frac{A_2(\eta)}{(\phi+1.5)^2} + \ldots + \frac{A_\gamma}{(\phi+1.5)^\gamma} + \frac{E_1(\eta)}{(\phi-\mu_1(1,\eta))} \\ &+ \frac{E_2(\eta)}{(\phi-\mu_2(1,\eta))}, \end{aligned}$$

where the functions $A_i(\eta), E_1(\eta)$, and $E_2(\eta)$ satisfy (3.67). Then,

$$C^+(r,\phi,\eta,\gamma) = \frac{A_1(\eta)}{(\phi+1.5)} + \frac{A_2(\eta)}{(\phi+1.5)^2} + \ldots + \frac{A_\gamma(\eta)}{(\phi+1.5)^\gamma},$$
$$C^-(r,\phi,\eta,\gamma) = \frac{E_1(\eta)}{(\phi-\mu_1(1,\eta))} + \frac{E_2(\eta)}{(\phi-\mu_2(1,\eta))}.$$

For $\eta = 0$ the numerator of (3.83) has a zero at the origin and a zero in the right half plane $Re(\phi) > 0$. We name these as $\mu_1(1,0) = 0$ and $\mu_2(1,0)$. From (3.80) we have

$$E[C^*] = \frac{(\beta-\alpha)}{\alpha}\frac{15c}{\mu_2(1,0)},$$

where c is the residue of

$$\frac{e^{-\beta\xi}}{\xi} \frac{(\mu_2(1,0) - \xi)}{(3-\xi)(5-\xi) - (15-4\xi)e^{-\beta\xi}}$$

at $\xi = 0$.

The values of the expected number of customers in steady state for various values of β are given in the following table.

β	ρ	$E\left[C^* \mid C_0 = 2\right]$
0.25	0.9375	7.969
0.23	0.825	3.049
0.2	0.75	1.125

The explicit expression for the transform $\int_0^\infty e^{-\eta t} E[C_t^* | C_0 = \gamma] dt$ can be obtained by first substituting the functions $[K^+(r,\phi,\eta)]^{-1}$, $C^+(r,\phi,\eta,\gamma)$, and $C^-(r,0,\eta,\gamma)$ into (3.9), and then by substituting the explicit expression for $Z(1,\phi,\eta,\gamma)$ into (3.36). We invert (3.36) numerically to get the expectation of number of customers in continuous time. We give a result in figure 3.4.

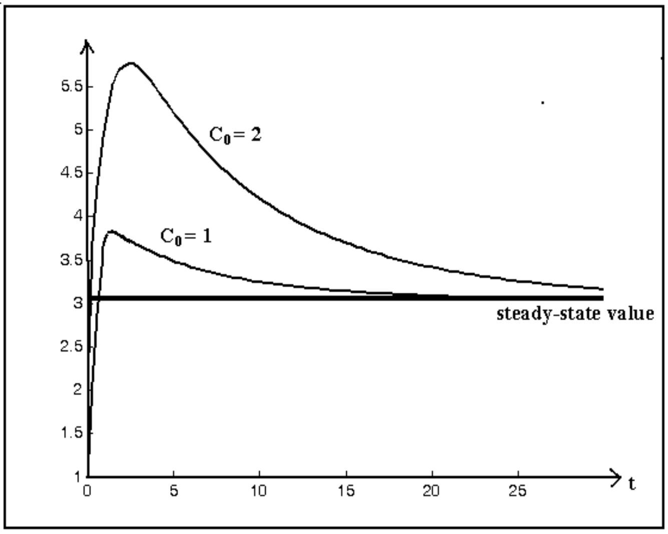

Figure 3.4: $E[C_t^* | C_0 = \gamma]$ with $\rho = 0.825$ for some values of the number of special customers γ.

3. *The system $E_2/M/1$*

   We suppose that the inter-arrival times have a E_2 distribution with Laplace-Stieltjes transform
$$A(\phi) = \frac{8^2}{(8+\phi)^2},$$
   and the service times have an exponential distribution with Laplace-Stieltjes transform
$$B(\phi) = \frac{6}{\phi+6}.$$
   The traffic intensity is $\rho = 2/3$. This system is a special case of the $GI/K_n/1$ system and the $K_m/G/1$ system as well, so we could follow the analysis in sub-section 3.7.1 or 3.7.2 in order to obtain the explicit expression for $Z(1, \phi, \eta, \gamma)$. We then can substitute the explicit expression into (3.36), and we invert (3.36) numerically to obtain the expectation of number of customers in continuous time.

   We give some results on $E[C_t^*|C_0 = \gamma]$ as t increases for some values of the number of special customers γ in figure 3.5.

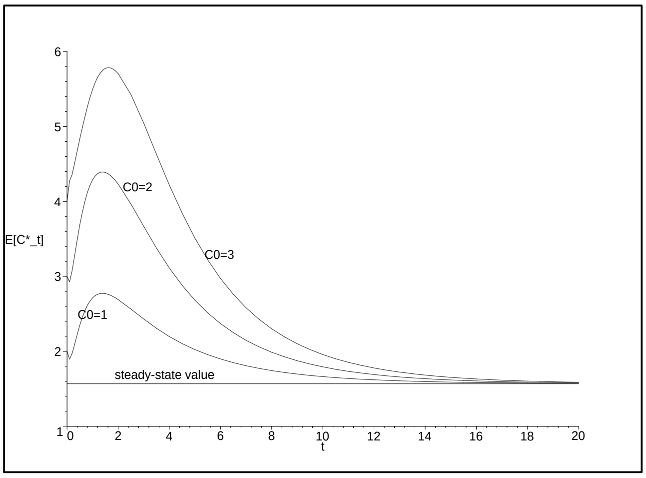

Figure 3.5: $E[C_t^*|C_0 = \gamma]$ for some values of the number of special customers γ

4. *The system $E_2/H_2/1$ with traffic intensity* $\rho = 0.175$

   We suppose that the inter-arrival times have an E_2 distribution with Laplace-Stieltjes transform
$$A(\phi) = \frac{6^2}{(6+\phi)^2},$$
   and the service times have an H_2 distribution with Laplace-Stieltjes transform
$$B(\phi) = \frac{600+35\phi}{(10+\phi)(60+\phi)}.$$
   The traffic intensity is $\rho = 0.175$. This system is a special case of the systems $GI/K_n/1$ system and $K_m/G/1$.

   We give some results on $E[C_t^*|C_0 = \gamma]$ as t increases for some values of the number of special customers γ in figure 3.6.

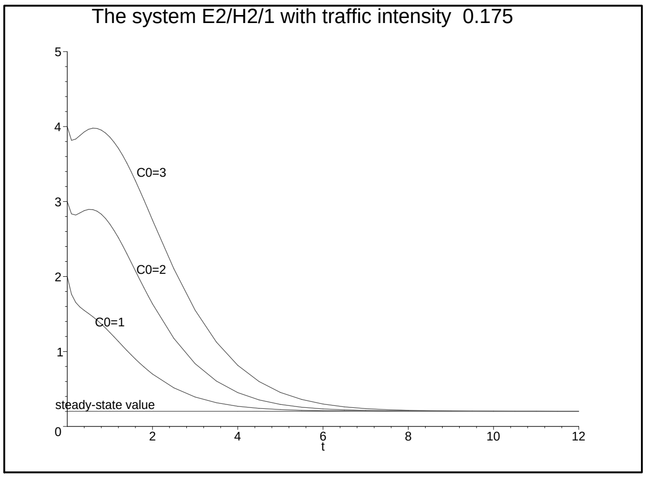

Figure 3.6: $E[C_t^*|C_0 = \gamma]$ for some values of the number of special customers γ

5. *The system $E_2/H_2/1$ with traffic intensity $\rho = 0.8$*

   We suppose that the inter-arrival times have a E_2 distribution with Laplace-Stieltjes transform

$$A(\phi) = \frac{6^2}{(6+\phi)^2},$$

   and the service times have an H_2 distribution with Laplace-Stieltjes transform

$$B(\phi) = \frac{15 + 4\phi}{(3+\phi)(5+\phi)}.$$

   The traffic intensity is $\rho = 0.8$. This system is an example of the $GI/K_n/1$ system and the $K_m/G/1$ system as well.

   We give some results on $E[C_t^*|C_0 = \gamma]$ as t increases for some values of the number of special customers γ in figure 3.7.

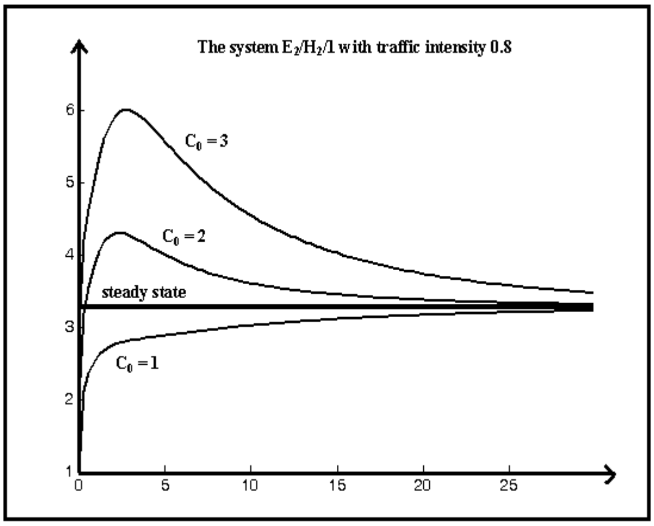

Figure 3.7: $E[C_t^*|C_0 = \gamma]$ for some values of the number of special customers γ

6. *The system $E_2/H_2/1$ with traffic intensity* $\rho = 0.9$

   We suppose that the inter-arrival times have an E_2 distribution with Laplace-Stieltjes transform
$$A(\phi) = \frac{6^2}{(6+\phi)^2},$$
   and the service times have an H_2 distribution with Laplace-Stieltjes transform
$$B(\phi) = \frac{20+6\phi}{(2+\phi)(10+\phi)}.$$
   The traffic intensity $\rho = 0.9$. This system is an example of the $GI/Kn/1$ system and the $K_m/G/1$ system as well.

   We give some results on $E[C_t^*|C_0 = \gamma]$ as t increases for some values of the number of special customers γ in figure 3.8.

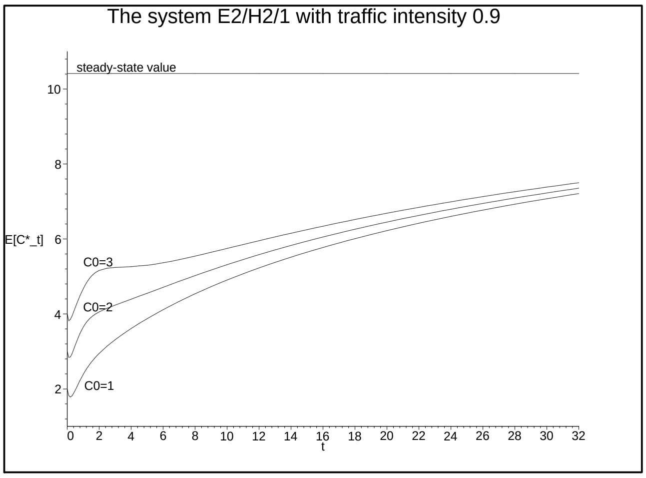

Figure 3.8: $E[C_t^*|C_0 = \gamma]$ for some values of the number of special customers γ

7. *The system $E_2/H_2/1$* with $\rho > 1$

   We suppose that the inter-arrival times have a E_2 distribution with Laplace-Stieltjes transform
$$A(\phi) = \frac{6^2}{(6+\phi)^2},$$
   and the service times have an H_2 distribution with Laplace-Stieltjes transform
$$B(\phi) = \frac{5+3\phi}{(1+\phi)(5+\phi)}.$$
   The traffic intensity $\rho = 1.8$. Not like the previous examples, here we have an unstable system.

Since, by assumption, the arriving customer at time $t = 0$ finds C_0 special customers in the system, the number of customers in the system at time t satisfies the relation

$$C_t^* = C_0 + 1 + N(t) - \int_0^t \mathbf{1}_{C_{s-}^*>0} dD(s), \tag{3.84}$$

where $N(t)$ and $D(t)$ denote the number of arrivals and the number of departures in $(0, t]$, respectively. Since $\rho > 1$, the probability to have an infinite busy cycle is positive, and hence after a finite t^* we will have an infinite busy cycle. It follows that for $t \geq t^*$, $\int_{t^*}^t \mathbf{1}_{C_{s-}^*>0} dD(s) = D^*(t)$, where $D^*(t)$ denotes the number of departures in the interval $[t^*, t]$, and, consequently, $\int_0^t \mathbf{1}_{C_{s-}^*>0} dD(s)$ is a delayed renewal process. Since the distributions of the inter-arrival times and the service times are non-lattice, then as a consequence, by applying the second order properties of a renewal process(see page 47 of Cox[20] or page page 158 of Asmussen[7]), we have for large t,

$$E[C_t^* | C_0 = \gamma] = \gamma + 1 + (1/\alpha - 1/\beta)t + \frac{E[A_1^2]}{2\alpha^2} - \frac{E[B_1^2]}{2\beta^2} + o(1). \tag{3.85}$$

Inverting (3.36) numerically we get the expectation of number of customers in continuous time. The result of the inversion, and its behavior with respect to (3.85) can be seen in figures 3.9 and 3.10.

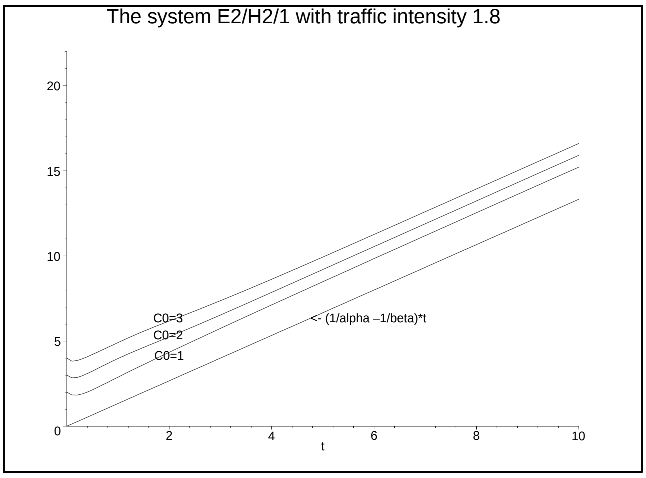

Figure 3.9: $E[C_t^* | C_0 = \gamma]$ for some values of the number of special customers γ, and the linear function $y(t) = (1/\alpha - 1/\beta)t$.

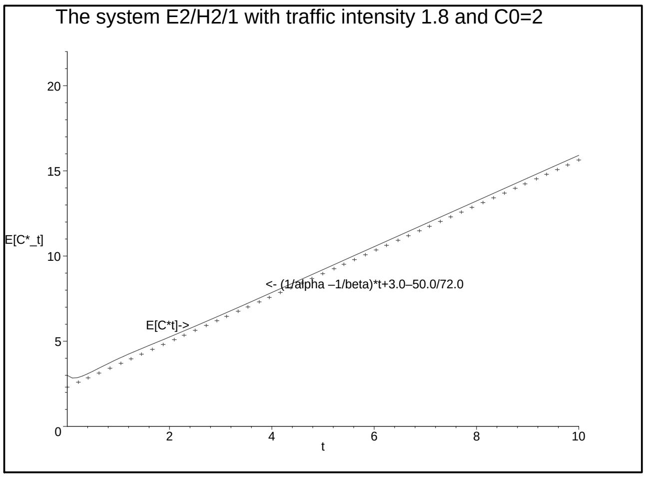

Figure 3.10: $E[C_t^*|C_0 = 2]$ and the linear function $y(t) = (1/\alpha - 1/\beta)t + C_0 + 1 + \frac{E[A_1^2]}{2\alpha^2} - \frac{E[B_1^2]}{2\beta^2}$.

Chapter 4

The $GI/H_m/s$ queue

4.1 Introduction

In this chapter we consider the many server queue $GI/H_m/s$, which is described as follows. Customers arrive at epochs $T_1, T_2, \ldots$ with $T_1 = 0$. The customer arriving at T_n is called the nth customer. The inter-arrival time between the $(n-1)$th and the nth customer is denoted by $A_n = T_n - T_{n-1}$, and the service time of the nth customer is denoted by $B_n, \quad n = 1, 2, \ldots$. There are s servers and the service discipline is first come, first served. We assume that $A_n, \quad n = 1, 2, \ldots$ are i.i.d with common distribution function F and mean α, $B_n, \quad n = 1, 2, \ldots$ are i.i.d with a common hyper-exponential distribution

$$G(x) = \begin{cases} \sum_{i=1}^{m} p_i(1 - exp(-b_i x)), & x \geq 0 \\ 0, & x < 0, \end{cases}$$

where $\sum_{i=1}^{m} p_i = 1, b_i > 0, b_i \neq b_j, i \neq j$. Furthermore, we assume that the sequences A_n and B_n are independent.

The customer who arrives at T_1 finds upon his arrival C_0 other customers in the system. We call these customers special customers, who we assume are numbered, describing their priority for service, from 1 up to C_0. The service of (some of) the special customers has just begun at T_1. Furthermore we assume that these customers have a common exponential service time with rate w, where $w = b_{\bar{j}}$ for a fixed $\bar{j}, 1 \leq \bar{j} \leq m$, and its distribution function is denoted by $I(x)$.

Let W_n be the actual waiting time of the nth customer and $W_{n,i}, \quad i = 1, 2, \ldots, s$, the service backlog or workload of the ith server just before the arrival of the nth customer, i.e., if the nth and subsequent customers would not enter the system then the ith server would become idle at time $T_n + W_{n,i}$. Since the queue discipline is first come, first served, we may assume that in front of each server there is a separate queue and that an arriving customer joins the queue of the server with the smallest workload. In the case when there are several servers with smallest workload, the arriving customer will select one of them at random. We also assume that this discipline holds for the special customers. Hence,

$$W_n = \min_{1 \leq i \leq s} W_{n,i}, \quad n = 1, 2, \ldots.$$

The study of this type of queue for $C_0 = 0$ has been done by de Smit [21, 22, 23]. In [21], using the Wiener-Hopf factorization method, the author studied some distributions of interest, such as the actual waiting time W_n, the queue length at server i at time T_n-, the number of customers in the system at time T_n-, and the number of customers in the system during the kth busy period. All the distributions are given in terms of Laplace-Stieltjes transforms.

In [23], the investigation in [21] is extended to the study of the system in continuous time. The results are expressions for the distributions of the virtual waiting time and the length of the busy period.

A numerical solution for the system $GI/H_2/s$ has been studied in [22]. The study is based on the Laplace-Stieltjes transforms of the distributions of interest derived in [21] and [23]. A numerical inversion algorithm is given, and some examples in this paper show that the algorithm has been successfully implemented to obtain good results.

In this chapter we try to extend the analysis in the papers mentioned above by assuming a non-zero initial number of customers. We are interested in those distributions, which for $C_0 = 0$ have been studied in [21] and [23], such as the distribution of actual waiting times, the distribution of virtual waiting times, the queue length distribution, and the distribution of the total number of customers. We consider the Markov process

$$\{(W_n, T_n, \mathbf{X}_n),\ n = 1, 2, \ldots\},$$

where $\mathbf{X}_n$ is the phase vector at T_n-, which will be defined in the next section. We refer to the derivation in [23] to obtain the system of Wiener-Hopf equations of the joint distribution of W_n, T_n and X_n. The system of equations we derive here is a generalization of what has been obtained in [21]. We solve this system of equations by first factorizing its symbol, and then decomposing a certain vector. We then obtain an explicit expression for the transform of the joint distribution of $\{(W_n, T_n, \mathbf{X}_n),\ n = 1, 2, \cdots\}$, which directly gives us the generating function of the waiting time of nth customer. The limiting distribution of the waiting time can be obtain by applying Abel's limit theorem, and the distribution of the waiting time of nth customer can be obtained by inverting the generating function numerically.

The transforms of the virtual waiting time, the queue length , and the total number of customers can be derived from the transform of of joint distribution of $\{(W_n, T_n, \mathbf{X}_n),$ $n = 1, 2, \cdots\}$. We show in sections 4.7, 4.8, 4.9 and 4.10, how the expressions for the transforms depend on the initial number of customers. The steady-state distributions of the distributions can be derived by applying Abel's limit theorem, and the time-dependent distributions can be obtained by inverting the transforms numerically. We apply the numerical inversion algorithm proposed in [3], and the results of the inversion can be found in section 4.11.

This chapter is organized as follow. In section 4.2 we recall some definitions and notations from [21]. In section 4.3 we derive the system of Wiener-Hopf equations, and then work out the factorization and the decomposition. The explicit expression for the Laplace-Stieltjes transform of the distribution of the actual waiting time can be found in section 4.5, and for the virtual waiting times can be found in section 4.6. The study of the distribution of the queue length at arrival epochs is done in section 4.7 and the distribution of total number of customers at arrival epochs is given in section 4.8. For the system in continuous time, the distributions of the queue length and the total number of customers are studied in section 4.9 and 4.10, respectively. Finally in section 4.11 some numerical examples for those distributions are given.

4.2 Notations and definitions

We denote the Laplace-Stieltjes transform of F by $A(\phi)$, i.e.,

$$A(\phi) = \int_0^\infty e^{-\phi x} dF(x), Re(\phi) \geq 0.$$

The mean of service time is denoted by β, thus

$$\beta = \sum_{i=1}^{m} \frac{p_i}{b_i}.$$

The relative traffic intensity ρ is defined by

$$\rho = \frac{\beta}{\alpha s}.$$

Suppose that $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ are two arbitrary $m-$ dimensional vectors. The inner product of vectors $\mathbf{x}$ and $\mathbf{y}$ is denoted by $\mathbf{xy} = x_1 y_1 + \ldots + x_m y_m$. We write $\mathbf{x} \leq \mathbf{y}$ if $x_1 \leq y_1, \cdots, x_m \leq y_m$. The vector $(x_1, \ldots, x_i \pm 1, x_{i+1}, \ldots, x_m)$ is denoted by $\mathbf{x} \pm \mathbf{1}_i$ and vector $(x_1 \pm y_1, \ldots, x_m \pm y_m)$ by $\mathbf{x} \pm \mathbf{y}$.

Let $\mathbf{R}_m^k$ be the class of m-dimensional vectors which have nonnegative integer components and for which $\mathbf{x1} = x_1 + \cdots + x_m = k$. $\mathbf{R}_m^k$ contains $\binom{m+k-1}{k}$ elements, see page 36 of Feller[26]. For brevity, we shall write $c(k)$ instead of $\binom{m+k-1}{k}$.

δ_{ij} is the Kronecker delta, i.e. $\delta_{ij} = 0, i \neq j, \delta_{jj} = 1$. $\mathbf{1}(A)$ is the indicator function of the event A. For real a we denote $a^+ = \max(0, a)$ and $a^- = \min(0, a)$.

If $\mathbf{M}$ is an $m \times n-$dimensional matrix, we denote by $\mathbf{M}_x$ the xth column of $\mathbf{M}$, and by $\mathbf{M}^x$ the xth row of $\mathbf{M}$.

We define

$$U_{n,i} = W_{n,i} - W_n, \quad i = 1, \ldots, s.$$

Note that an arriving customer joins the queue with smallest workload, we see that $U_{n,i}$, if non-zero, is the remaining service time at $T_n + W_n$ of the last customer who joined the queue of server i before T_n. If this customer is of type j, this remaining service time is exponentially distributed with parameter b_j. We then say that at time T_n- server i is in phase j. Let $X_{n,i}$ be the number of servers that at T_n- are in phase i; the vector $\mathbf{X}_n = (X_{n,1}, \ldots, X_{n,m})$ is called the phase vector of the system at time T_n- and we see that $\mathbf{X}_n \in \bigcup_{k=0}^{s-1} \mathbf{R}_m^k$. Given $\mathbf{X}_n$, those $U_{n,i}$ that are strictly positive and W_n are mutually independent, and $\{(W_n, T_n, \mathbf{X}_n), n = 1, 2, \ldots\}$ is a vector Markov process.

Let $\mathbf{X}_n^*$ be the phase vector at time T_n, i.e. $\mathbf{X}_n^* = \mathbf{X}_n + \mathbf{1}_i$ if the nth customer is of type i. Observe that $\mathbf{X}_n^* \in \bigcup_{k=1}^{s} \mathbf{R}_m^k$. Let us define

$$V_n = \min{}^{(2)}(U_{n,1}, \ldots, U_{n,s}, B_n),$$

where $\min^{(2)}(x_1, \ldots, x_k)$ is the smallest but one element of $(x_1, \ldots, x_k)$. Let $\mathbf{Y}_n$ be the phase vector corresponding to $(U_{n,1}, \ldots, U_{n,s}, B_n)$ that are not equal to 0 or V_n. If $\mathbf{X}_n^* = \mathbf{x}$ with $\mathbf{x1} < s$ then $V_n = W_n = 0$ and $\mathbf{Y}_n = \mathbf{x}$; if $\mathbf{X}_n^* = \mathbf{x}$ with $\mathbf{x1} = s$ it follows from Lemma 1 of Appendix 1 in de Smit [21] that $\mathbf{Y}_n = \mathbf{x} - \mathbf{1}_j$ with probability $x_j b_j / xb$.

For readability, in this chapter we write all vectors and matrices in bold or with bar accent.

4.3 Wiener-Hopf factorization

In this section we will derive the system of Wiener-Hopf equations governing the Laplace-Stieltjes transform of the joint distribution of W_n, T_n, and $\mathbf{X}_n$. To obtain a solution for this system we use Wiener-Hopf factorization and a decomposition.

We recall that the first customer upon his arrival finds C_0 other customers in the system. There are two possible cases for C_0 to be considered.

1. $C_0 < s$.
   This means that at T_1 at least one server is idle so that $W_1 = 0$, and we assume $\mathbf{X}_1 = (\bar{x}_1, \ldots, \bar{x}_m)$. It is clear that for this case $\mathbf{X}_1 \in \bigcup_{k=0}^{s-1} \mathbf{R}_m^k$.

2. $C_0 \geq s$.
   This means that at T_1 all servers are busy so that $W_1 > 0$. Since by assumption the service times of the special customers are i.i.d and have a common exponential distribution with rate w, the distribution of W_1 will be the convolution of exponential distributions with rate w. To illustrate this, let us consider the example described by figure 4.1.

   In this example, we denote by $\tilde{B}_i$ the service time of the ith special customer. We see that
   $$E(exp(-\phi W_1)) = \left(\frac{w}{\phi + w}\right)^2,$$
   and
   $$X_{1,\bar{j}} = 4,$$
   noting that $w = b_{\bar{j}}$.

In general, our assumption on the service times of the special customers leads to the following condition.

Condition 4.3.1
For $Re(\phi) \geq 0$,

$$E(exp(-\phi W_1)) = \left(\frac{w}{\phi + w}\right)^a,$$

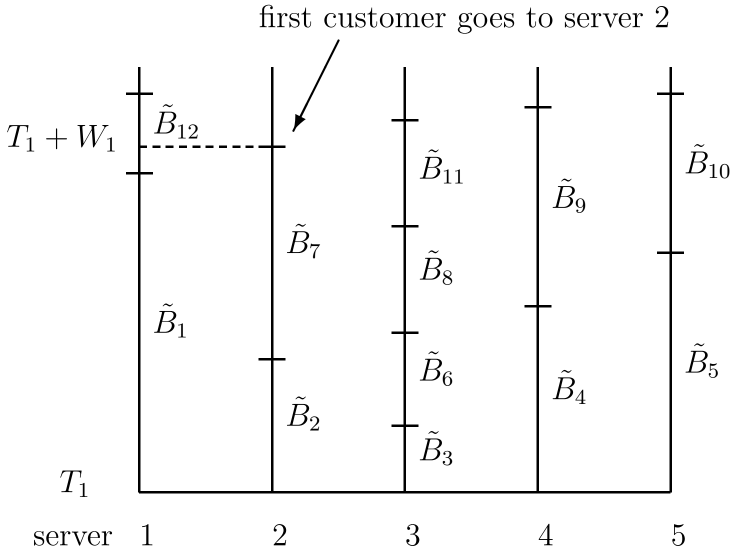

Figure 4.1: A system with $s = 5$ and $C_0 = 12$

$$\textit{where } a = \begin{cases} 0 & , \textit{for } C_0 < s, \\ \textit{a positive integer} & , \textit{otherwise.} \end{cases}$$

Observe that for $C_0 \geq s$, the busy servers at time T_1+W_1 are serving special customers only. This means that

$$X_{1,i} = \begin{cases} 0 & , i \neq \bar{j}, \\ \#\text{ busy servers at time } T_1+W_1 & , i = \bar{j}, \end{cases}$$

and, as a consequence, we can assume $\mathbf{X}_1 = (0,\ldots,0,\bar{x}_{\bar{j}},0,\cdots,0) = \bar{x} \in \bigcup_{k=0}^{s-1} \mathbf{R}_m^k$.

Remark. The results we obtain in this chapter will depend on C_0 through its value γ.

We define for $\gamma = 0,1,\ldots$, $|r| < 1$, $Re(\eta) \geq 0$ or $\gamma = 0,1,\ldots$, $|r| \leq 1$, $Re(\eta) > 0$,

$$Z(r,\eta,\gamma;\mathbf{x}) = \sum_{n=1}^{\infty} r^n E(exp(-\eta T_n)1(\mathbf{X}_n = \mathbf{x})|C_0 = \gamma), \quad \mathbf{x} \in \bigcup_{k=0}^{s-2} \mathbf{R}_m^k,$$

$$Z^*(r,\eta,\gamma;\mathbf{x}) = \sum_{i=1}^{m} p_i Z(r,\eta,\gamma;\mathbf{x}-\mathbf{1}_i), \quad \mathbf{x} \in \mathbf{R}_m^{s-1}, \tag{4.1}$$

$$\mathcal{Z}_n(\eta,\phi,\gamma;\mathbf{x}) = E(exp(-\phi W_n - \eta T_n)1(\mathbf{X}_n = \mathbf{x})|C_0 = \gamma), n = 1,2,\cdots, \mathbf{x} \in \mathbf{R}_m^{s-1},$$

$$\mathcal{Z}(r,\eta,\phi,\gamma;\mathbf{x}) = \sum_{n=1}^{\infty} r^n \mathcal{Z}_n(\eta,\phi,\gamma;\mathbf{x}), \quad \mathbf{x} \in \mathbf{R}_m^{s-1},$$

and

$$D(r,\eta,\phi,\gamma;\mathbf{x}) = \sum_{n=1}^{\infty} r^{n+1} E(exp(\phi[W_n + V_n - A_{n+1}]^- - \eta T_n)1(\mathbf{Y}_n = \mathbf{x})|C_0 = \gamma),$$

$$\mathbf{x} \in \mathbf{R}_m^{s-1}, Re(\phi) \geq 0.$$

With the assumptions on $\mathbf{X}_1$ and W_1 we obtain the system of Wiener-Hopf equations given in the following theorem. This is a generalization of Theorem 2.1 in de Smit[21].

Let

$$\binom{\mathbf{x}}{\mathbf{y};\mathbf{z}} = \prod_{i=1}^{m} \frac{x_i!}{y_i! z_i! (x_i - y_i - z_i)!}$$

and let $\mathbf{b} = (b_1, b_2, \cdots, b_m)$.

Theorem 4.3.1

For $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbf{R}_m^k, k = 0, 1, \ldots, s-2$;

$$\begin{aligned}
&Z(r,\eta,\gamma;\mathbf{x}) \\
&= r\prod_{i=1}^{m} \delta_{\bar{x}_i, x_i} + rA(\eta + \mathbf{xb}) \sum_{i=1}^{m} p_i Z(r,\eta,\gamma;\mathbf{x} - \mathbf{1}_i) \\
&\quad + \sum_{l=k+1}^{s-2} \sum_{\mathbf{y}\in\mathbf{R}_m^l} \sum_{\boldsymbol{\nu}\leq\mathbf{y}-\mathbf{x}} \binom{\mathbf{y}}{\mathbf{x};\boldsymbol{\nu}} (-1)^{\boldsymbol{\nu}\mathbf{1}} rA(\eta + \mathbf{xb} + \nu\mathbf{b}) \sum_{i=1}^{m} p_i Z(r,\eta,\gamma;\mathbf{y} - \mathbf{1}_i) \\
&\quad + \sum_{\mathbf{y}\in\mathbf{R}_m^{s-1}} \sum_{\boldsymbol{\nu}\leq\mathbf{y}-\mathbf{x}} \binom{\mathbf{y}}{\mathbf{x};\boldsymbol{\nu}} (-1)^{\boldsymbol{\nu}\mathbf{1}} D(r,\eta,\mathbf{xb} + \nu\mathbf{b},\gamma;\mathbf{y}),
\end{aligned} \tag{4.2}$$

and for $\mathbf{x} \in \mathbf{R}_m^{s-1}, Re(\phi) = 0$,

$$\begin{aligned}
&\mathcal{Z}(r,\eta,\phi,\gamma;\mathbf{x}) \left[1 - r\sum_{j=1}^{m} p_j \frac{(x_j+1)b_j}{\phi + \mathbf{xb} + b_j} A(\eta - \phi)\right] \\
&= rA(\eta-\phi) \sum_{i=1}^{m} p_i Z(r,\eta,\gamma;\mathbf{x} - \mathbf{1}_i) \\
&\quad + rA(\eta-\phi) \sum_{j=1}^{m} \sum_{i=1, i\neq j}^{m} p_i \mathcal{Z}(r,\eta,\phi,\gamma;\mathbf{x} + \mathbf{1}_j - \mathbf{1}_i) \frac{(x_j+1)b_j}{\phi + \mathbf{xb} + b_j} \\
&\quad + r\mathcal{Z}_1(\phi,\gamma;\mathbf{x}) + \mathcal{Z}(r,\eta,0,\gamma;\mathbf{x}) - D(r,\eta,-\phi,\gamma;\mathbf{x}),
\end{aligned} \tag{4.3}$$

while for $\mathbf{x} \in \mathbf{R}_m^{s-1}$,

$$\mathcal{Z}(r,\eta,0,\gamma;\mathbf{x}) = D(r,\eta,\mathbf{xb},\gamma;\mathbf{x}), \tag{4.4}$$

and the vector $\bar{\mathcal{Z}}_1(\phi,\gamma)$, by letting $\mathbf{X}_1 = (\bar{x}_1, \ldots, \bar{x}_m)$, has elements

$$\mathcal{Z}_1(\phi,\gamma;\mathbf{x}) = \begin{cases} 0 & , \text{ if } \gamma < s-1, \\ \prod_{i=1}^{m} \delta_{\bar{x}_i, x_i} & , \text{ if } \gamma = s-1, \\ \prod_{i=1}^{m} \delta_{\bar{x}_i, x_i} \left(\frac{w}{\phi + w}\right)^a & , \text{ if } \gamma \geq s, \end{cases}$$

where a is a positive integer less than s, as described in Condition 4.3.1.

Proof. We can follow the proof of Theorem 2.1 in de Smit [21] to obtain the equations (4.2), (4.3) and (4.4). It remains to determine the expression for $\mathcal{Z}_1(\phi, \gamma; \mathbf{x})$, $\mathbf{x} \in \mathbf{R}_m^{s-1}$.

For $\mathbf{x} \in \mathbf{R}_m^{s-1}$,

$$\mathcal{Z}_1(\phi, \gamma; \mathbf{x}) = E(\exp(-\phi W_1)\mathbf{1}(\mathbf{X}_1 = \mathbf{x})|C_0 = \gamma).$$

For $\gamma = 0$, $\mathbf{X}_1 \in \bigcup_{k=0}^{s-2} \mathbf{R}_m^k$ so that $\mathbf{1}(\mathbf{X}_1 = \mathbf{x}) = 0$. It follows that $\mathcal{Z}_1(\phi, \gamma; \mathbf{x}) = 0$, so that for $\gamma = 0$ the equation (4.3) is precisely the same as the equation (2.2) in de Smit[23]. For $\gamma = s - 1$, $W_1 = 0$ so that $\mathcal{Z}_1(\phi, \gamma; \mathbf{x}) = 1$ if and only if $\mathbf{x} = \mathbf{X}_1$. For $\gamma \geq s$, we use the condition 4.3.1 so that $\mathcal{Z}_1(\phi, \gamma; \mathbf{x}) = \prod_{i=1}^m \delta_{\bar{x}_i, x_i} \left(\frac{w}{\phi+w}\right)^a$.

■

Define the $c(s-1) \times c(s-1)$-dimensional matrix $\mathbf{H}(r, \eta, \phi)$ with elements $H_{\mathbf{x},\mathbf{y}}(r, \eta, \phi), \mathbf{x}, \mathbf{y} \in R_m^{s-1}$, as

$$\begin{aligned} H_{x,x}(r, \eta, \phi) &= 1 - rA(\eta - \phi) \sum_{j=1}^m p_j \frac{(x_j + 1)b_j}{\phi + \mathbf{x}\mathbf{b} + b_j} \\ H_{x,x+1_j-1_i}(r, \eta, \phi) &= -rA(\eta - \phi) p_i \frac{(x_j + 1)b_j}{\phi + \mathbf{x}\mathbf{b} + b_j}, \quad i \neq j, x_i > 0, \\ H_{x,y}(r, \eta, \phi) &= 0, \qquad \text{otherwise.} \end{aligned} \tag{4.5}$$

Let $\bar{\mathcal{Z}}(r, \eta, \phi, \gamma), \bar{\mathcal{D}}(r, \eta, \phi, \gamma), \mathbf{Z}^*(r, \eta, \gamma)$ and $\bar{\mathcal{Z}}_1(\phi, \gamma)$ be the $c(s-1)-$dimensional column vectors with elements $\mathcal{Z}(r, \eta, \phi, \gamma; \mathbf{x}), \mathcal{D}(r, \eta, \phi, \gamma; \mathbf{x}), Z^*(r, \eta, \gamma; \mathbf{x})$ and $\mathcal{Z}_1(\phi, \gamma; \mathbf{x})$ respectively. Equation (4.3) can then be written in the matrix form:

$$\begin{aligned} &\mathbf{H}(r, \eta, \phi)\bar{\mathcal{Z}}(r, \eta, \phi, \gamma) \\ =&rA(\eta - \phi)\mathbf{Z}^*(r, \eta, \gamma) + r\bar{\mathcal{Z}}_1(\phi, \gamma) + \bar{\mathcal{Z}}(r, \eta, 0, \gamma) - \bar{\mathcal{D}}(r, \eta, -\phi, \gamma). \end{aligned} \tag{4.6}$$

The system (4.6) can be solved by factorizing its symbol $\mathbf{H}(r, \eta, \phi)$ and then decomposing the vector $r\mathbf{H}^{-1}(r, \eta, \phi)\bar{\mathcal{Z}}_1(\phi, \gamma)$. The factorization is similar to the one in [21]. The result is given in Theorem 4.3.3, and after that we discuss the decomposition. First, we recall a theorem from [21] that is needed for the factorization.

Theorem 4.3.2
Let $N(r, \eta)$ be the total order of the zeros of $\det \mathbf{H}(r, \eta, \phi)$ *in the left half-plane $Re(\phi) < 0$. If $0 < |r| < 1$ and $Re(\eta) \geq 0$, or $0 < |r| \leq 1$ and $Re(\eta) > 0$, or $r = 1, \eta = 0$ and $\rho < 1$ and if some conditions on $\mathbf{H}(r, \eta, \phi)$ are satisfied, then $N(r, \eta) = c(s)$. For $|r| < 1 \det \mathbf{H}(r, \eta, \phi) \neq 0$ on $Re(\phi) = 0$; for $\rho < 1 \det \mathbf{H}(1, \eta, \phi)$ has a simple zero at $\phi = 0$ and has no zero elsewhere on the imaginary axis.*

Proof. See [21]. ■

We denote the zeros of $\det \mathbf{H}(r, \eta, \phi)$ in the left half-plane $Re(\phi) < 0$ by $\mu_{\mathbf{x}}(r, \eta)$, $\mathbf{x} \in \mathbf{R}_m^s$. These zeros are continuous functions of r in $[0, 1]$. We impose the following condition.

Condition 4.3.2
For $(|r| < 1,\ Re(\eta) \geq 0)$ *or* $(|r| \leq 1,\ Re(\eta) > 0)$, *all zeros* $\mu_{\mathbf{x}}(r,\eta)$ *are of order 1.*

This condition and the conditions mentioned in the Theorem 4.3.2 are almost always satisfied (see [21] for more explanation).

For $\mathbf{y} \in \mathbf{R}^s_m$ let $\mathbf{B_y}$ be a non-zero $c(s-1)-$dimensional column vector satisfying

$$\mathbf{H}(r,\eta,\mu_{\mathbf{y}}(r,\eta))\mathbf{B_y} = 0, \tag{4.7}$$

and let $\mathbf{B}$ be the $c(s-1) \times c(s)-$dimensional matrix whose column vectors are the $\mathbf{B_y}$. Moreover we introduce the following matrices:

the $c(s) \times c(s)-$dimensional matrix $\mathbf{L}$ with elements

$$\mathbf{L_{x,y}} = \sum_{\{i|x_i>0\}} p_i \mathbf{B}_{\mathbf{x}-\mathbf{1}_i,\mathbf{y}} \frac{1}{\mu_{\mathbf{y}}(r,\eta) + \mathbf{xb}}, \quad \mathbf{x} \in \mathbf{R}^s_m, \mathbf{y} \in \mathbf{R}^s_m;$$

the $c(s) \times c(s-1)-$dimensional matrix $\mathbf{M}$ with elements

$$\mathbf{M_{x,y}} = \sum_{i=1}^{m} p_i \delta_{\mathbf{x},\mathbf{y}+\mathbf{1}_i}, \quad \mathbf{x} \in \mathbf{R}^s_m, \mathbf{y} \in \mathbf{R}^{s-1}_m;$$

the $c(s) \times c(s)-$dimensional matrix $\mathbf{J}(\eta,\phi)$ with elements

$$\mathbf{J_{x,x}}(\eta,\phi) = \frac{1}{\phi - \mu_{\mathbf{x}}(r,\eta)}, \quad \mathbf{x} \in \mathbf{R}^s_m,$$

$$\mathbf{J_{x,y}}(\eta,\phi) = 0, \quad \mathbf{x} \in \mathbf{R}^s_m, \mathbf{y} \in \mathbf{R}^s_m, \mathbf{x} \neq \mathbf{y};$$

and the $c(s-1) \times c(s-1)-$identity matrix $\mathbf{I}$.

We shall assume that the following condition holds.

Condition 4.3.3
$\det \mathbf{L} \neq 0$.

If Condition 4.3.3 holds the $c(s) \times c(s-1)-$dimensional matrix $\mathbf{C}$ is determined by the set of linear equations

$$\mathbf{LC} = \mathbf{M}. \tag{4.8}$$

Note that the matrices $\mathbf{B}, \mathbf{L}$, and $\mathbf{C}$ depend on r and η. For notational convenience, we suppress this dependence. We define the $c(s-1) \times c(s-1)$ dimensional matrices $\mathbf{K}(r,\eta,\phi)$ and $\mathbf{H}^-(r,\eta,\phi)$ by

$$\mathbf{K}(r,\eta,\phi) = \mathbf{I} + \mathbf{BJ}(\eta,\phi)\mathbf{C}, \tag{4.9}$$

and

$$\mathbf{H}^-(r,\eta,\phi) = \mathbf{H}(r,\eta,\phi)\mathbf{K}(r,\eta,\phi) = \mathbf{H}(r,\eta,\phi) + \mathbf{H}(r,\eta,\phi)\mathbf{BJ}(\eta,\phi)\mathbf{C}. \tag{4.10}$$

The following theorem gives the factors of matrix $\mathbf{H}(r,\eta,\phi)$, which exist if a number of conditions hold. de Smit [21] argues that these conditions are almost always satisfied and their exclusion does not cause any serious practical restriction, because they can be approximated arbitrarily closely by cases for which the conditions are satisfied.

Theorem 4.3.3
If conditions 4.3.2 and 4.3.3 and other conditions on $\det \mathbf{H}(r, \phi)$ *hold then*

1.
$$\det \mathbf{K}(r, \eta, \phi) = \prod_{\mathbf{x} \in \mathbf{R}_m^s} \left(\frac{\phi + \mathbf{x}\mathbf{b}}{\phi - \mu_{\mathbf{x}}(r, \eta)} \right),$$
$\det \mathbf{H}^-(r, \eta, \phi)$ *is bounded away from* 0 *for* $Re(\phi) \leq 0$;

2. *For* $Re(\phi) = 0$,
$$\mathbf{H}(r, \eta, \phi) = \mathbf{H}^-(r, \eta, \phi)\mathbf{H}^+(r, \eta, \phi)$$
where $\mathbf{H}^+(r, \phi, \eta) = \mathbf{K}(r, \eta, \phi)^{-1}$ *satisfies property* A^+ *and is non-singular in* $Re(\phi) > 0$, *and* $\mathbf{H}^-(r, \phi, \eta)$ *satisfies property* A^- *and is non-singular in* $Re(\phi) < 0$.

Proof. See [21]. ■

With the factorization above we can write (4.6), for $Re(\phi) = 0$, as

$$\begin{aligned}\mathbf{H}^+(r, \eta, \phi)\bar{\mathcal{Z}}(r, \eta, \phi, \gamma) =& \mathbf{H}^-(r, \eta, \phi)^{-1}[rA(\eta - \phi)\mathbf{Z}^*(r, \eta, \gamma) + \bar{\mathcal{Z}}(r, \eta, 0, \gamma)] \\ &+ \mathbf{H}^-(r, \eta, \phi)^{-1}[-\bar{\mathcal{D}}(r, \eta, -\phi, \gamma) + r\bar{\mathcal{Z}}_1(\phi, \gamma)].\end{aligned} \tag{4.11}$$

From Theorem 4.3.3 we see that $H^-(r, \eta, \phi)^{-1}$ satisfies A^+. Moreover, from the expression for $\bar{\mathcal{Z}}_1(\phi, \gamma)$ in Theorem 4.3.1 we see that for $\gamma < s$, the vector $\bar{\mathcal{Z}}_1(\phi, \gamma)$ has the same property as $\mathbf{H}^-(r, \eta, \phi)^{-1}$. It follows that the vector $r\mathbf{H}^-(r, \eta, \phi)^{-1}\bar{\mathcal{Z}}_1(\phi, \gamma)$ in (4.11) satisfies A^+. For $\gamma \geq s$, $\bar{\mathcal{Z}}_1(\phi, \gamma)$ does not satisfy A^-, so that we have to decompose the vector $r\mathbf{H}^-(r, \eta, \phi)^{-1}\bar{\mathcal{Z}}_1(\phi, \gamma)$, i.e. we need to determine $c(s-1)$-dimensional column vectors $\bar{\mathcal{Z}}_1^+(r, \eta, \phi, \gamma)$ and $\bar{\mathcal{Z}}_1^-(r, \eta, \phi, \gamma)$ with elements $\mathcal{Z}_1^+(r, \eta, \phi, \gamma; \mathbf{x})$ and $\mathcal{Z}_1^-(r, \eta, \phi, \gamma; \mathbf{x}), \mathbf{x} \in \mathbf{R}_m^{s-1}$, respectively, such that

$$r\mathbf{H}^-(r, \eta, \phi)^{-1}\bar{\mathcal{Z}}_1(\phi, \gamma) = \bar{\mathcal{Z}}_1^+(r, \eta, \phi, \gamma) + \bar{\mathcal{Z}}_1^-(r, \eta, \phi, \gamma), \tag{4.12}$$

where $\bar{\mathcal{Z}}_1^+(r, \eta, \phi, \gamma)$ satisfies A^+ and $\bar{\mathcal{Z}}_1^-(r, \eta, \phi, \gamma)$ satisfies A^-.

Although the decomposition is needed only for the case ($\gamma \geq s$ and $\mathbf{X}_1 \in \mathbf{R}_m^{s-1}$), in the following we give the expressions for $\bar{\mathcal{Z}}_1^+(r, \eta, \phi, \gamma)$ and $\bar{\mathcal{Z}}_1^-(r, \eta, \phi, \gamma)$ for all cases, in order to formulate the general solution for (4.6).

Notice that for $\gamma \geq s - 1$, the elements $\mathcal{Z}_1(\phi, \gamma; x)$ of $\bar{\mathcal{Z}}_1(\phi, \gamma)$ are equal to zero except for $\mathbf{x} = \bar{x}$, where $\bar{x}$ is the phase vector at time T_1. It follows that the '$\mathbf{x}$th' element of $r\mathbf{H}^-(r, \eta, \phi)^{-1}\bar{\mathcal{Z}}_1(\phi, \gamma)$ for $\gamma \geq s - 1$ is given by

$$r\mathbf{H}^-_{\mathbf{x},\bar{x}}(r, \eta, \phi)^{-1}\mathcal{Z}_1(\phi, \gamma; \bar{x}).$$

Let

$$h^{(j)}(r, \eta) = \frac{1}{j!}\frac{d^j}{d\phi^j}\mathbf{H}^-_{\mathbf{x},\bar{x}}(r, \eta, \phi)^{-1}\Big|_{\phi=-w} \text{ and } h^{(0)}(r, \eta) = \mathbf{H}^-_{\mathbf{x},\bar{x}}(r, \eta, -w)^{-1}.$$

For $\gamma < s - 1$, the vector $r\mathbf{H}^-(r, \eta, \phi)^{-1}\bar{\mathcal{Z}}_1(\phi, \gamma)$ is the vector zero.

Lemma 4.3.1
For $\gamma \geq s$ and $\mathbf{x} \in \mathbf{R}_m^{s-1}$, the elements of the column vectors $\bar{\mathcal{Z}}_1^+(r,\eta,\phi,\gamma)$ and $\bar{\mathcal{Z}}_1^-(r,\eta,\phi,\gamma)$ are given by

$$\mathcal{Z}_1^-(r,\eta,\phi,\gamma;\mathbf{x}) = \frac{rw^a}{(\phi+w)^a}\left[\mathbf{H}_{\mathbf{x},\bar{x}}^-(r,\eta,\phi)^{-1} - \sum_{j=0}^{a-1} h^{(j)}(r,\eta)(\phi+w)^j\right],$$

$$\mathcal{Z}_1^+(r,\eta,\phi,\gamma;\mathbf{x}) = rw^a \sum_{j=0}^{a-1} \frac{h^{(j)}(r,\eta)}{(\phi+w)^{a-j}},$$

for $\gamma = s-1$ and $\mathbf{x} \in \mathbf{R}_m^{s-1}$,

$$\mathcal{Z}_1^-(r,\eta,\phi,\gamma;\mathbf{x}) = r\mathbf{H}_{\mathbf{x},\bar{x}}^-(r,\eta,\phi)^{-1},$$
$$\mathcal{Z}_1^+(r,\eta,\phi,\gamma;\mathbf{x}) = 0,$$

and for $\gamma < s-1$ and $\mathbf{x} \in \mathbf{R}_m^{s-1}$,

$$\mathcal{Z}_1^-(r,\eta,\phi,\gamma;\mathbf{x}) = 0,$$
$$\mathcal{Z}_1^+(r,\eta,\phi,\gamma;\mathbf{x}) = 0,$$

With the factorization given by Theorem 4.3.3 and the decomposition in Lemma 4.3.1 we obtain the solution of the system (4.3) that is given in the following theorem.

Theorem 4.3.4
If all conditions mentioned before hold, then for $0 < |r| < 1, Re(\phi) \geq 0$,

$$\begin{aligned}&\bar{\mathcal{Z}}(r,\eta,\phi,\gamma)\\ =&\mathbf{K}(r,\eta,\phi)\mathbf{K}(r,\eta,0)^{-1}\bar{\mathcal{Z}}(r,\eta,0,\gamma) + \mathbf{K}(r,\eta,\phi)[\bar{\mathcal{Z}}_1^+(r,\eta,\phi,\gamma) - \bar{\mathcal{Z}}_1^+(r,\eta,0,\gamma)].\end{aligned} \tag{4.13}$$

Proof. From (4.11), (4.12) and (4.3.1) we have for $Re(\phi) = 0$,

$$\begin{aligned}&\mathbf{H}^+(r,\eta,\phi)\bar{\mathcal{Z}}(r,\eta,\phi,\gamma) - \bar{\mathcal{Z}}_1^+(r,\eta,\phi,\gamma)\\ =&\mathbf{H}^-(r,\eta,\phi)^{-1}[rA(\eta-\phi)\mathbf{Z}^*(r,\eta,\gamma) + \bar{\mathcal{Z}}(r,\eta,0,\gamma) - \bar{\mathcal{D}}(r,\eta,-\phi,\gamma)]\\ &+ \bar{\mathcal{Z}}_1^-(r,\eta,\phi,\gamma).\end{aligned} \tag{4.14}$$

The left-hand side of (4.14) satisfies A^+; the right-hand side of (4.14) satisfies A^-. By analytic continuation we can define an entire function that is equal to the left-hand side for $Re(\phi) \geq 0$ and equal to the right-hand side for $Re(\phi) \leq 0$. But this entire function is bounded and hence a constant by Liouville's theorem. So for $Re(\phi) \leq 0$,

$$\begin{aligned}&\mathbf{H}^-(r,\eta,\phi)^{-1}[rA(\eta-\phi)\mathbf{Z}^*(r,\eta,\gamma) + \bar{\mathcal{Z}}(r,\eta,0,\gamma) - \bar{\mathcal{D}}(r,\eta,-\phi,\gamma)]\\ &+ \bar{\mathcal{Z}}_1^-(r,\eta,\phi,\gamma) = \mathbf{H}^+(r,\eta,0)\bar{\mathcal{Z}}(r,\eta,0,\gamma) - \bar{\mathcal{Z}}_1^+(r,\eta,0,\gamma).\end{aligned} \tag{4.15}$$

Moreover, for $Re(\phi) \geq 0$,

$$\mathbf{H}^+(r,\eta,\phi)\bar{\mathcal{Z}}(r,\eta,\phi,\gamma) - \bar{\mathcal{Z}}_1^+(r,\eta,\phi,\gamma) = \mathbf{H}^+(r,\eta,0)\bar{\mathcal{Z}}(r,\eta,0,\gamma) - \bar{\mathcal{Z}}_1^+(r,\eta,0,\gamma), \tag{4.16}$$

which yields

$$\begin{aligned}\bar{\mathcal{Z}}(r,\eta,\phi,\gamma) =&\mathbf{H}^+(r,\eta,\phi)^{-1}\mathbf{H}^+(r,\eta,0)\bar{\mathcal{Z}}(r,\eta,0,\gamma)\\ &+\mathbf{H}^+(r,\eta,\phi)^{-1}\left[\bar{\mathcal{Z}}_1^+(r,\eta,\phi,\gamma)-\bar{\mathcal{Z}}_1^+(r,\eta,0,\gamma)\right]\end{aligned}$$

and using part 2 of the Theorem 4.3.3 we get (4.13). This completes the proof. ■

Equation (4.13) gives us an expression for $\bar{\mathcal{Z}}(r,\eta,\phi,\gamma)$ that depends on $\bar{\mathcal{Z}}(r,\eta,0,\gamma)$. To find an expression for $\bar{\mathcal{Z}}(r,\eta,0,\gamma)$, we let

$$\mathbf{S}(r,\eta,\phi)=\mathbf{H}^-(r,\eta,\phi)\mathbf{H}^+(r,\eta,0),\quad Re(\phi)\le 0.$$

From (4.15) we have for $Re(\phi)\le 0$,

$$\begin{aligned}\bar{\mathcal{D}}(r,\eta,-\phi,\gamma) =&rA(\eta-\phi)\mathbf{Z}^*(r,\eta,\gamma)+\bar{\mathcal{Z}}(r,\eta,0,\gamma)-\mathbf{S}(r,\eta,\phi)\bar{\mathcal{Z}}(r,\eta,0,\gamma)\\ &+\mathbf{H}^-(r,\eta,\phi)\left[\bar{\mathcal{Z}}_1^-(r,\eta,\phi,\gamma)+\bar{\mathcal{Z}}_1^+(r,\eta,0,\gamma)\right],\end{aligned}\tag{4.17}$$

so that for $\mathbf{x}\in\mathbf{R}_m^k, k=0,1,\cdots,s-1;\mathbf{y}\in\mathbf{R}_m^{s-1}$,

$$\begin{aligned}&\mathcal{D}(r,\eta,\mathbf{xb},\gamma;\mathbf{y})\\ =&rA(\eta+\mathbf{xb})Z^*(r,\eta,\gamma;\mathbf{y})+\mathcal{Z}(r,\eta,0,\gamma;\mathbf{y})\\ &-\sum_{\mathbf{w}\in\mathbf{R}_m^{s-1}}\mathbf{S}_{\mathbf{y},\mathbf{w}}(r,\eta,-\mathbf{xb})\mathcal{Z}(r,\eta,0,\gamma;\mathbf{w})\\ &+\sum_{\mathbf{w}\in\mathbf{R}_m^{s-1}}\mathbf{H}^-_{\mathbf{y},\mathbf{w}}(r,\eta,-\mathbf{xb})\left[\mathcal{Z}_1^-(r,\eta,-\mathbf{xb},\gamma;\mathbf{w})+\mathcal{Z}_1^+(r,\eta,0,\gamma;\mathbf{w})\right].\end{aligned}\tag{4.18}$$

Choosing $\mathbf{x}=\mathbf{y}$ and using (4.4) we have for $\mathbf{y}\in\mathbf{R}_m^{s-1}$,

$$\begin{aligned}&\sum_{\mathbf{w}\in\mathbf{R}_m^{s-1}}\mathbf{S}_{\mathbf{y},\mathbf{w}}(r,\eta,-\mathbf{yb})\mathcal{Z}(r,\eta,0,\gamma;\mathbf{w})\\ =&rA(\eta+\mathbf{yb})Z^*(r,\eta,\gamma;\mathbf{y})+\mathcal{Z}_2(r,\eta,-\mathbf{yb},\gamma;\mathbf{y}),\end{aligned}\tag{4.19}$$

where

$$\begin{aligned}&\mathcal{Z}_2(r,\eta,\phi,\gamma;\mathbf{y})\\ =&\sum_{\mathbf{w}\in\mathbf{R}_m^{s-1}}\mathbf{H}^-_{\mathbf{y},\mathbf{w}}(r,\eta,\phi)\left[\mathcal{Z}_1^-(r,\eta,\phi,\gamma;\mathbf{w})+\mathcal{Z}_1^+(r,\eta,0,\gamma;\mathbf{w})\right],\mathbf{y}\in\mathbf{R}_m^{s-1}.\end{aligned}\tag{4.20}$$

Define the $c(s-1)\times c(s-1)-$dimensional matrix $\mathbf{Q}(r,\eta)$ by

$$\mathbf{Q}_{\mathbf{x},\mathbf{y}}(r,\eta)=\mathbf{H}^-_{\mathbf{x},\mathbf{y}}(r,\eta,-\mathbf{xb}),\quad \mathbf{x}\in\mathbf{R}_m^{s-1},\mathbf{y}\in\mathbf{R}_m^{s-1}.$$

Condition 4.3.4
For $(|r|<1,\ Re(\eta)\ge 0)$ *or* $(|r|\le 1,\ Re(\eta)>0)$, $\det\mathbf{Q}(r,\eta)\ne 0$.

We assume that Condition 4.3.4 is satisfied. Let

$$\mathbf{R}(r,\eta) = \mathbf{Q}(r,\eta)\mathbf{H}^{+}(r,\eta,0).$$

Then, for $\mathbf{x} \in \mathbf{R}_m^{s-1}$,

$$\begin{aligned} &\mathcal{Z}(r,\eta,0,\gamma;\mathbf{x}) \\ &= \sum_{\mathbf{y}\in\mathbf{R}_m^{s-1}} (\mathbf{R}(r,\eta))^{-1}_{\mathbf{x},\mathbf{y}} \left[rA(\eta+\mathbf{yb})Z^{*}(r,\eta,\gamma;\mathbf{y}) + \mathcal{Z}_2(r,\eta,-\mathbf{yb},\gamma;\mathbf{y})\right]. \end{aligned} \tag{4.21}$$

If we substitute (4.21) into (4.13) we get an explicit expression for $\mathcal{Z}(r,\eta,\phi,\gamma;\mathbf{x})$, $\mathbf{x} \in \mathbf{R}_m^{s-1}$. In the following, we will derive an explicit expression for $Z(r,\eta,\gamma;\mathbf{x})$, $\mathbf{x} \in \bigcup_{k=0}^{s-2} \mathbf{R}_m^k$ by first determining expression for $\mathcal{D}(r,\eta,-\phi,\gamma;\mathbf{x})$, $\mathbf{x} \in \mathbf{R}_m^{s-1}$.

Define

$$\mathbf{G}(r,\eta,\phi) = (\mathbf{I} - \mathbf{S}(r,\eta,\phi))\mathbf{R}(r,\eta)^{-1},$$

then from (4.18) we have for $Re(\phi) \le 0, \mathbf{x} \in \mathbf{R}_m^{s-1}$,

$$\begin{aligned} &\mathcal{D}(r,\eta,-\phi,\gamma;\mathbf{x}) \\ =&rA(\eta-\phi)\mathbf{Z}^{*}(r,\eta,\gamma;\mathbf{x}) + \mathcal{Z}_2(r,\eta,\phi,\gamma;\mathbf{x}) \\ &+ \sum_{\mathbf{y}\in\mathbf{R}_m^{s-1}} \mathbf{G}_{\mathbf{x},\mathbf{y}}(r,\eta,\phi)\left[rA(\eta+\mathbf{yb}))\mathbf{Z}^{*}(r,\eta-\mathbf{yb},\gamma;\mathbf{y}) + \mathcal{Z}_2(r,\eta,\phi,\gamma;\mathbf{y})\right]. \end{aligned} \tag{4.22}$$

Substitution into (4.2) yields for $\mathbf{x} \in \mathbf{R}_m^k, k = 0,1,\cdots,s-2$;

$$\begin{aligned} Z(r,\eta,\gamma;\mathbf{x}) = &\sum_{l=(k-1)^{+}}^{s-3} \sum_{\mathbf{y}\in\mathbf{R}_m^l} Z(r,\eta,\gamma;\mathbf{y}) r \sum_{i=1}^{m} p_i c_1(\eta;\mathbf{x},\mathbf{y}+\mathbf{1}_i) \\ &+ \sum_{\mathbf{y}\in\mathbf{R}_m^{s-2}} Z(r,\eta,\gamma;\mathbf{y}) r \sum_{i=1}^{m} p_i[c_1(\eta;\mathbf{x},\mathbf{y}+\mathbf{1}_i) + c_2(r,\eta;\mathbf{x},\mathbf{y}+\mathbf{1}_i)] \\ &+ c_3(r,\eta,\gamma;\mathbf{x}), \end{aligned} \tag{4.23}$$

where

$$c_1(\eta;\mathbf{x},\mathbf{y}) = \sum_{\nu\le\mathbf{y}-\mathbf{x}} \binom{\mathbf{y}}{\mathbf{x};\nu} (-1)^{\nu\mathbf{1}} A(\eta+\mathbf{xb}+\nu\mathbf{b})$$

$$c_2(r,\eta;\mathbf{x},\mathbf{y}) = \sum_{\mathbf{z}\in\mathbf{R}_m^{s-1}} \sum_{\nu\le\mathbf{z}-\mathbf{x}} \binom{\mathbf{z}}{\mathbf{x};\nu} (-1)^{\nu\mathbf{1}} \mathbf{G}_{\mathbf{z},\mathbf{y}}(r,\eta,-\mathbf{xb}-\nu\mathbf{b})A(\eta+\mathbf{yb}),$$

and

1. For $\gamma \ge s$ and $\mathbf{X}_1 \in \mathbf{R}_m^{s-1}$,

$$\begin{aligned} c_3(r,\eta,\gamma;\mathbf{x}) = &\sum_{\mathbf{z}\in\mathbf{R}_m^{s-1}} c_2(r,\eta,\mathbf{x},\mathbf{z})\mathcal{Z}_2(r,\eta,-\mathbf{zb},\gamma;\mathbf{z})/A(\eta+\mathbf{zb}) \\ &+ \sum_{\mathbf{y}\in\mathbf{R}_m^{s-1}} \sum_{\nu\le\mathbf{y}-\mathbf{x}} \binom{\mathbf{y}}{\mathbf{x};\nu} (-1)^{\nu\mathbf{1}} \mathcal{Z}_2(r,\eta,-\mathbf{xb}-\nu\mathbf{b};\mathbf{y}), \end{aligned}$$

2. For $\gamma < s-1$ or ($\gamma \geq s$ and $\mathbf{X}_1 \notin \mathbf{R}_m^{s-1}$), $\bar{\mathcal{Z}}_1^-(r,\eta,\phi,\gamma)$ and $\bar{\mathcal{Z}}_1^+(r,\eta,\phi,\gamma)$ turn out be zero and this implies $\mathcal{Z}_2(r,\eta,\phi,\gamma;\mathbf{y}) = 0, \forall \mathbf{y} \in \mathbf{R}_m^{s-1}$, so that
$$c_3(r,\eta,\gamma;\mathbf{x}) = r\prod_{i=1}^{m} \delta_{\bar{x}_i,x_i},$$
3. For $\gamma = s-1$, Lemma 4.3.1 implies that $\mathcal{Z}_2(r,\eta,\phi,\gamma;\mathbf{y}) = r\prod_{i=1}^{m}\delta_{\bar{x}_i,y_i}$. Notice that in this case $\mathbf{X}_1 \in \mathbf{R}_m^{s-1}$, so that $r\prod_{i=1}^{m}\delta_{\bar{x}_i,x_i} = 0$. We then have
$$\begin{aligned} c_3(r,\eta,\gamma;\mathbf{x}) =& r\sum_{\mathbf{y}\in\mathbf{R}_m^{s-1}}\sum_{\nu\leq\mathbf{y}-\mathbf{x}}\binom{\mathbf{y}}{\mathbf{x};\nu}(-1)^{\nu\mathbf{1}}\mathbf{G}_{\mathbf{y},\bar{x}}(r,\eta,-\mathbf{x}\mathbf{b}-\nu\mathbf{b}) \\ &+ r\sum_{\nu\leq\bar{x}-\mathbf{x}}\binom{\bar{x}}{\mathbf{x};\nu}(-1)^{\nu\mathbf{1}}. \end{aligned}$$

Let the $\binom{m+s-2}{s-2} \times \binom{m+s-2}{s-2}$-dimensional matrix $\mathbf{T}(r,\eta)$ with elements
$$\mathbf{T}_{\mathbf{x},\mathbf{y}}(r,\eta), \quad \mathbf{x} \in \bigcup_{k=0}^{s-2}\mathbf{R}_m^k, \quad \mathbf{y} \in \bigcup_{k=0}^{s-2}\mathbf{R}_m^k,$$
are defined by
$$\mathbf{T}_{\mathbf{x},\mathbf{y}}(r,\eta) = \begin{cases} r\sum_{i=1}^{m} p_i c_1(\eta;\mathbf{x},\mathbf{y}+\mathbf{1}_i) & \text{, for } \mathbf{x}\in\mathbf{R}_m^k, \\ & \mathbf{y}\in\mathbf{R}_m^l, \text{with} \\ & (1\leq k\leq s-2; \\ & k-1\leq l\leq s-3) \\ & \text{or} \\ & k=0; 0\leq l\leq s-3, \\ r\sum_{i=1}^{m} p_i[c_1(\eta;\mathbf{x},\mathbf{y}+\mathbf{1}_i)+c_2(r,\eta;\mathbf{x},\mathbf{y}+\mathbf{1}_i)] & \text{, for } \mathbf{x}\in\mathbf{R}_m^k, \\ & k=0,\dots,s-2; \\ & \mathbf{y}\in\mathbf{R}_m^{s-2}, \\ 0 & \text{, otherwise.} \end{cases}$$

Let $\mathbf{I}$ be the $\binom{m+s-2}{s-2} \times \binom{m+s-2}{s-2}$ identity matrix, $\mathbf{Z}(r,\eta,\gamma)$ be the $\binom{m+s-2}{s-2}$-dimensional column vector with elements $Z(r,\eta,\gamma;\mathbf{x})$, $\mathbf{x} \in \bigcup_{k=0}^{s-2}\mathbf{R}_m^k$, and $\mathbf{c_3}(r,\eta,\gamma)$ be the $\binom{m+s-2}{s-2}$-dimensional column vector with elements $c_3(r,\eta;\mathbf{x})$, $\mathbf{x} \in \bigcup_{k=0}^{s-2}\mathbf{R}_m^k$. With these definitions (4.23) becomes
$$(\mathbf{I}-\mathbf{T}(r,\eta))\mathbf{Z}(r,\eta,\gamma) = \mathbf{c_3}(r,\eta,\gamma), \tag{4.24}$$

which is a generalization of the system (3.19) in de Smit [21].

We have studied the system of equations of the time-dependent transforms $Z(r,\eta,\gamma;\mathbf{x})$, $\mathbf{x}\in\bigcup_{k=0}^{s-2}\mathbf{R}_m^k$ and $Z(r,\eta,0,\gamma;\mathbf{x}),\mathbf{x}\in\mathbf{R}_m^{s-1}$, which we derive in (4.24) and (4.21). In the rest of this chapter all time-dependent probability distributions of interest are derived in terms of these transforms. We need to impose the following condition in order to have an unique solution of the systems (4.24) and (4.21).

Condition 4.3.5
For $(|r|<1,\ Re(\eta)\geq 0)$ *or* $(|r|\leq 1,\ Re(\eta)>0)$, $\det(\mathbf{I}-\mathbf{T}(r,\eta))\neq 0$.

We end this section with stating a theorem that will be used later in section 4.6. The theorem is a generalization to Theorem 1 in de Smit [23].

Theorem 4.3.5
For $\gamma\geq s,\mathbf{X}_1\in\mathbf{R}_m^{s-1}$, *we have for* $Re(\phi)\geq 0$ *and* $Re(\eta)>0$

$$\begin{aligned}&\sum_{n=1}^{\infty}E(exp(-\phi(W_n+V_n)-\eta T_n)1(\mathbf{Y}_n=\mathbf{y})|C_0=\gamma)\\&=\sum_{n=1}^{m}p_iZ(1,\eta,\gamma;\mathbf{y}-\mathbf{1}_i)+\sum_{\mathbf{x}\in\mathbf{R}_m^s}\left(\frac{V_1(1,\eta,\gamma;\mathbf{x})}{(\phi-\mu_{\mathbf{x}}(1,\eta))}-\sum_{i=1}^{a}\frac{a_i(\mathbf{x})V_2(1,\eta;\mathbf{x})}{(\phi+w)^{a-i+1}}\right)\\&\quad\cdot\left(\sum_{j=1}^{m}(y_j+1)b_j\mathbf{L}_{\mathbf{y}+\mathbf{1}_j,\mathbf{x}}\right),\quad \mathbf{y}\in\mathbf{R}_m^{s-1},\end{aligned}\tag{4.25}$$

and

$$\begin{aligned}&\sum_{i=1}^{\infty}E(exp(-\phi(W_n+V_n)-\eta T_n)|C_0=\gamma)\\&=\sum_{k=1}^{s-2}\sum_{\mathbf{y}\in\mathbf{R}_m^k}\sum_{i=1}^{m}p_iZ(1,\eta,\gamma;\mathbf{y}-\mathbf{1}_i)\\&\quad+\sum_{\mathbf{x}\in\mathbf{R}_m^s}\left(\frac{V_1(1,\eta,\gamma;\mathbf{x})}{(\phi-\mu_{\mathbf{x}}(1,\eta))}-\sum_{i=1}^{a}\frac{a_i(\eta,\mathbf{x})V_2(1,\eta;\mathbf{x})}{(\phi+w)^{a-i+1}}\right)\\&\quad\cdot\left(\frac{1}{A(\eta-\mu_{\mathbf{x}}(1,\eta))}\sum_{\mathbf{y}\in\mathbf{R}_m^{s-1}}\mathbf{B}_{\mathbf{y},\mathbf{x}}\right),\end{aligned}\tag{4.26}$$

where

$$\begin{aligned}a_i(\eta,x)&=\frac{1}{(i-1)!}\lim_{\phi\to -w}\frac{\partial^{i-1}}{\partial\phi^{i-1}}\frac{1}{(\phi-\mu_{\mathbf{x}}(1,\eta))}\\&=\frac{(-1)^{i-1}}{(i-1)!}(-w-\mu_{\mathbf{x}}(1,\eta))^{-i},\end{aligned}\tag{4.27}$$

and $V_1(r,\eta;\mathbf{x})$ *and* $V_2(r,\eta;\mathbf{x})$, $\mathbf{x}\in\mathbf{R}_m^s$, *respectively, are defined as*

$$V_1(r,\eta,\gamma;\mathbf{x}) = [\mathbf{CK}(r,\eta,0)^{-1}\mathbf{Z}(r,\eta,0,\gamma)]_{\mathbf{x}} - [\mathbf{C}\bar{\mathcal{Z}}_1^{+}(r,\eta,0,\gamma)]_{\mathbf{x}} + \frac{V_2(1,\eta;\mathbf{x})}{(\mu_{\mathbf{x}}(1,\eta)+w)^a}, \tag{4.28}$$

$$V_2(r,\eta;\mathbf{x}) = rw^a \sum_{\mathbf{v}\in\mathbf{R}_m^{s-1}} \mathbf{C}_{\mathbf{x},\mathbf{v}} \left[\mathbf{H}^{-}(r,0,-w)^{-1}\right]_{\mathbf{v},\bar{x}}, \tag{4.29}$$

otherwise, we have for $\mathbf{y}\in\mathbf{R}_m^{s-1}$, $Re(\phi)\geq 0$ *and* $Re(\eta)>0$,

$$\begin{aligned}&\sum_{n=1}^{\infty} E(exp(-\phi(W_n+V_n)-\eta T_n)1(\mathbf{Y}_n=\mathbf{y})|C_0=\gamma)\\ &=\sum_{n=1}^{m} p_i Z(1,\eta,\gamma;\mathbf{y}-\mathbf{1}_i) + \sum_{\mathbf{x}\in\mathbf{R}_m^s} \frac{V(1,\eta,\gamma;\mathbf{x})}{(\phi-\mu_{\mathbf{x}}(1,\eta))}\left(\sum_{j=1}^{m}(y_j+1)b_j\mathbf{L}_{\mathbf{y}+\mathbf{1}_j,\mathbf{x}}\right),\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}&\sum_{i=1}^{\infty} E(exp(-\phi(W_n+V_n)-\eta T_n)|C_0=\gamma)\\ &=\sum_{k=1}^{s-1}\sum_{\mathbf{y}\in\mathbf{R}_m^k}\sum_{i=1}^{m} p_i Z(1,\eta,\gamma;\mathbf{y}-\mathbf{1}_i)\\ &\quad+\sum_{\mathbf{x}\in\mathbf{R}_m^s}\frac{V(1,\eta,\gamma;\mathbf{x})}{(\phi-\mu_{\mathbf{x}}(1,\eta))}\left(\frac{1}{A(\eta-\mu_{\mathbf{x}}(1,\eta))}\sum_{\mathbf{y}\in\mathbf{R}_m^{s-1}}\mathbf{B}_{\mathbf{y},\mathbf{x}}\right),\end{aligned} \tag{4.31}$$

where $\mathbf{B}_{\mathbf{x},\mathbf{y}}$ *is defined in (4.7).*

Proof. The proof of (4.25) and (4.26) is similar to the proof of Theorem 1 in [23] and will be omitted. We use Theorem 16.20 in Apostol [5] for the derivation of $a_i(\eta,\mathbf{x})$.

For $\gamma<s$ or ($\gamma\geq s$ and $\mathbf{X}_1\notin\mathbf{R}_m^{s-1}$), by our choice of $\mathcal{Z}_1^{+}(r,\eta,\phi)$ and $\mathcal{Z}_1^{-}(r,\eta,\phi)$ we have

$$\frac{V_1(1,\eta,\gamma;\mathbf{x})}{(\phi-\mu_{\mathbf{x}}(1,\eta))} - \sum_{i=1}^{a}\frac{a_i(\mathbf{x})V_2(1,\eta;\mathbf{x})}{(\phi+w)^{a-i+1}} = \frac{V(1,\eta,\gamma;\mathbf{x})}{(\phi-\mu_{\mathbf{x}}(1,\eta))},$$

where the column vector $V(r,\eta,\gamma)$ with components $V(r,\eta,\gamma,\mathbf{x}),\mathbf{x}\in\mathbf{R}_m^s$ is defined as

$$\mathbf{V}(r,\eta,\gamma) = \mathbf{CK}(r,\eta,0)^{-1}\bar{\mathcal{Z}}(r,\eta,0,\gamma). \tag{4.32}$$

In this case, equation (4.25) becomes equation (4.30), and equation (4.26) becomes equation (4.31).

■

The equations (4.30) and (4.31) are precisely the same as equations (2.4) and (2.5) in Theorem 1 in [23], where the $GI/H_m/s$ system with $\gamma=0$ is studied.

4.4 Steady state results

Before we study some distributions of interest, in this section we study the phase vectors and some related transforms in steady state.

For $\rho < 1$ we define for $\mathbf{x} \in \mathbf{R}_m^k, k = 0, 1, \cdots, s-2$;

$$X(\mathbf{x}) = \lim_{n\to\infty} P(\mathbf{X}_n = \mathbf{x})$$

and for $\mathbf{x} \in \mathbf{R}_m^{s-1}$,

$$\begin{aligned}
\mathcal{X}^*(\mathbf{x}) &= \lim_{n\to\infty} \sum_{i=1}^m p_i P(\mathbf{X}_n = \mathbf{x} - \mathbf{1}_i) \\
\mathcal{X}(\phi; \mathbf{x}) &= \lim_{n\to\infty} E(exp(-\phi W_n) 1(\mathbf{X}_n = \mathbf{x})) \\
\Delta(\phi; \mathbf{x}) &= \lim_{n\to\infty} E(exp(\phi [W_n + V_n - A_{n+1}]^-) 1(\mathbf{Y}_n = \mathbf{x})).
\end{aligned}$$

Let $\bar{\mathcal{X}}^*, \bar{\mathcal{X}}(\phi)$ and $\bar{\Delta}(\phi)$ be the $c(s-1)$-dimensional column vectors with elements $\mathcal{X}^*(\mathbf{x})$, $\mathcal{X}(\phi; \mathbf{x})$, and $\Delta(\phi; \mathbf{x})$ respectively, and write $\mathbf{H}(\phi) = \mathbf{H}(1, 0, \phi), \mathbf{K}(\phi) = \mathbf{K}(1, 0, \phi)$. From (4.2) and

$$\sum_{k=0}^{s-2} \sum_{\mathbf{x}\in\mathbf{R}_m^k} Z(r, \eta, \gamma; \mathbf{x}) + \sum_{\mathbf{x}\in\mathbf{R}_m^{s-1}} \mathcal{Z}(r, \eta, 0, \gamma; \mathbf{x}) = \frac{r}{1 - rA(\eta)}, \tag{4.33}$$

we have for $\mathbf{x} \in \mathbf{R}_m^k, k = 0, 1, \cdots, s-2$;

$$X(\mathbf{x}) = \lim_{r\uparrow 1} (1 - r) Z(r, 0, 0; \mathbf{x}) = \lim_{\eta\downarrow 0} (1 - A(\eta)) Z(1, \eta, 0; \mathbf{x}),$$

and for $\mathbf{x} \in \mathbf{R}_m^{s-1}$,

$$\mathcal{X}^*(\mathbf{x}) = \lim_{r\uparrow 1} (1 - r) \sum_{i=1}^m p_i Z(r, 0, 0; \mathbf{x} - \mathbf{1}_i) = \lim_{\eta\downarrow 0} (1 - A(\eta)) Z(1, \eta, 0; \mathbf{x} - \mathbf{1}_i),$$

$$\mathcal{X}(\phi; \mathbf{x}) = \lim_{r\uparrow 1} (1 - r) \mathcal{Z}(r, 0, \phi, 0; \mathbf{x}) = \lim_{\eta\downarrow 0} (1 - A(\eta)) \mathcal{Z}(1, \eta, \phi, 0; \mathbf{x}),$$

and

$$\Delta(\phi; \mathbf{x}) = \lim_{r\uparrow 1} (1 - r) \mathcal{D}(r, 0, \phi, 0; \mathbf{x}) = \lim_{\eta\downarrow 0} (1 - A(\eta)) \mathcal{D}(1, \eta, \phi, 0; \mathbf{x}),$$

so that (4.6) implies

$$\mathbf{H}(\phi) \bar{\mathcal{X}}(\phi) = A(-\phi) \mathcal{X}^* + \bar{\mathcal{X}}(0) - \bar{\Delta}(-\phi).$$

From (4.13) we obtain the solution of (4.3) in steady state.

Theorem 4.4.1

If $\rho < 1$ and appropriate generalizations of Conditions 3.3 - 3.6 as well as Condition 3.6' in [21] hold for $r = 1$ then for $Re(\phi) \geq 0$,

$$\bar{\mathcal{X}}(\phi) = \mathbf{K}(\phi) \mathbf{K}(0)^{-1} \bar{\mathcal{X}}(0). \tag{4.34}$$

Proof. It is clear that

$$\lim_{r\uparrow 1}(1-r)\mathbf{K}(r,\eta,\phi)[\mathcal{Z}_1^+(r,\eta,\phi,0)-\mathcal{Z}_1^+(r,\eta,0,0)]=0.$$

Then by applying Abel's theorem into (4.13) we get (4.34).

■

Let $\mathbf{X}$ be the $\binom{m+s-2}{s-2}$-dimensional column vector with elements $X(\mathbf{x}), \mathbf{x} \in \bigcup_{k=0}^{s-2} \mathbf{R}_m^k$. Equation (4.24) becomes

$$(\mathbf{I}-\mathbf{T}(1,0))\mathbf{X}=0, \tag{4.35}$$

and we have the normalizing condition

$$\sum_{k=0}^{s-2}\sum_{\mathbf{x}\in\mathbf{R}_m^k} X(\mathbf{x}) + \sum_{\mathbf{x}\in\mathbf{R}_m^{s-1}} \mathcal{X}(0;\mathbf{x}) = 1. \tag{4.36}$$

If we apply Abel's theorem to the function in (4.21) then we have for $\mathbf{x}\in\mathbf{R}_m^{s-1}$,

$$\mathcal{X}(0;\mathbf{x}) = \sum_{\mathbf{y}\in\mathbf{R}_m^{s-1}} \mathbf{R}(1,0)^{-1}_{\mathbf{x},\mathbf{y}} A(\mathbf{y}\mathbf{b})\mathcal{X}^*(\mathbf{x}). \tag{4.37}$$

So that we can write (4.36) as

$$\sum_{k=0}^{s-3}\sum_{\mathbf{x}\in\mathbf{R}_m^k} X(\mathbf{x}) + \sum_{\mathbf{x}\in\mathbf{R}_m^{s-2}} X(\mathbf{x}) \left[1+\sum_{\mathbf{y}\in\mathbf{R}_m^{s-1}}\sum_{i=1}^{m} p_i\mathbf{R}(1,0)^{-1}_{\mathbf{y},\mathbf{x}+\mathbf{1}_i} A(\mathbf{x}\mathbf{b}+b_i)\right] = 1. \tag{4.38}$$

We then impose the following condition.

Condition 4.4.1
The matrix $\mathbf{I}-\mathbf{T}(1,0)$ *has rank* $\binom{m+s-2}{s-2}-1$.

Since equation (4.38) is not a linear combination of the equations in the system (4.35), then, if the Condition 4.4.1 holds, system (4.35) plus the equation (4.38) has rank

$$\binom{m+s-2}{s-2},$$

so that $X(\mathbf{x}), \mathbf{x} \in \bigcup_{k=0}^{s-2}\mathbf{R}_m^k$ can be obtained from (4.35) and (4.36). Moreover, $\mathcal{X}(0;\mathbf{x})$, $\mathbf{x}\in\mathbf{R}_m^{s-1}$ can be obtained from equation (4.37).

4.5 The actual waiting time

In this section we derive the distribution of the actual waiting of the nth customer and the distribution of the actual waiting time in steady state.

For $|r| < 1, Re(\phi) \geq 0$, by conditioning on the phase vector at time 0 we have

$$\begin{aligned}\sum_{n=1}^{\infty} r^n E(exp(-\phi W_n)|C_0 = \gamma) &= \sum_{k=0}^{s-2} \sum_{\mathbf{x}\in\mathbf{R}_m^k} Z(r,0,\gamma;\mathbf{x}) + \sum_{\mathbf{x}\in\mathbf{R}_m^{s-1}} \mathcal{Z}(r,0,\phi,\gamma;\mathbf{x}) \\ &= \frac{r}{1-r} + \sum_{\mathbf{x}\in\mathbf{R}_m^{s-1}} [\mathcal{Z}(r,0,\phi,\gamma;\mathbf{x}) - \mathcal{Z}(r,0,0,\gamma;\mathbf{x})].\end{aligned}$$

The expression for the function $\mathcal{Z}(r,0,\phi,\gamma;\mathbf{x})$ or $\mathcal{Z}(r,0,0,\gamma;\mathbf{x})$ can be obtained from (4.13) by setting η (and ϕ) equal to 0. Then by using (4.9), after some simple calculations, we have for $\gamma \geq s, \mathbf{X}_1 \in \mathbf{R}_m^{s-1}$, $|r| < 1$, and $Re(\phi) \geq 0$,

$$\begin{aligned}&\sum_{n=1}^{\infty} r^n E(exp(-\phi W_n)|C_0 = \gamma) \\ =&\frac{r}{1-r} - \left(\bar{\mathcal{Z}}_1^+(r,0,0,\gamma)\right)^t \mathbf{1} + \sum_{\mathbf{x}\in\mathbf{R}_m^s} \frac{e_{\mathbf{x}}(r,0,\gamma)\phi + f_{\mathbf{x}}(r)}{\phi - \mu_{\mathbf{x}}(r,0)} \\ &+ w^a \frac{\left(\bar{\mathcal{Z}}_1^+(r,0,0)\right)^t \mathbf{1}}{(\phi+w)^a} + \sum_{i=1}^{a} \frac{g_i(r)}{(\phi+w)^{a-i+1}},\end{aligned} \tag{4.39}$$

where the $c(s)$-dimensional column vectors $\mathbf{e}(r,\eta,\gamma)$ and $\mathbf{f}(r)$ with elements $e_{\mathbf{x}}(r,\eta)$ and $f_{\mathbf{x}}(r)$, respectively, are given by

$$e(r,\eta,\gamma) = \mathbf{E}(r,\eta)\mathbf{K}(r,\eta,0)^{-1}\bar{\mathcal{Z}}(r,\eta,0,\gamma) \tag{4.40}$$

$$f_x(r) = \left[\frac{1}{(w+\mu_{\mathbf{x}}(r,0))^a} - \frac{1}{w^a}\right] \mathbf{B}_{\mathbf{x}}^t \mathbf{1} V_2(r,0;\mathbf{x}), \tag{4.41}$$

with the $c(s) \times c(s-1)$-dimensional matrix $\mathbf{E}(r,\eta)$ is defined as

$$\mathbf{E}_{x,y}(r,\eta) = \frac{1}{\mu_{\mathbf{x}}(r,\eta)} \mathbf{C}_{\mathbf{x},\mathbf{y}} \sum_{\mathbf{z}\in\mathbf{R}_m^{s-1}} \mathbf{B}_{\mathbf{z},\mathbf{x}}, \tag{4.42}$$

the $l-$dimensional column vector $\mathbf{g}(r)$ with elements $g_i(r)$ is given by

$$g_i(r) = \sum_{\mathbf{y}\in\mathbf{R}_m^s} a_i(0,\mathbf{y}) \mathbf{B}_{\mathbf{y}}^t \mathbf{1} V_2(r,0;\mathbf{y}),$$

where the function $V_2(r,0;\mathbf{y})$ is defined in (4.29), and the function $a_i(\eta,\mathbf{y})$ is defined in (4.27).

For $\gamma < s$ or $(\gamma \geq s, \mathbf{X}_1 \notin \mathbf{R}_m^{s-1})$, it turns out that $f_{\mathbf{x}}(r) = 0, \mathbf{x} \in \mathbf{R}_m^s$, and $g_i(r) = 0$, for $i = 1, 2, \cdots, l$, so that the equation (4.39) becomes

$$\sum_{n=1}^{\infty} r^n E(exp(-\phi W_n)|C_0 = \gamma) = \frac{r}{1-r} + \sum_{\mathbf{x}\in\mathbf{R}_m^s} \frac{e_{\mathbf{x}}(r,0,\gamma)\phi}{\phi - \mu_{\mathbf{x}}(r,0)}, \tag{4.43}$$

which for $\gamma = 0$ is identical to equation (4.1) in [21].

If $\rho < 1$ let W be the actual waiting time in steady state. Then by applying Abel's theorem to (4.39) or (4.43) we have for $Re(\phi) \geq 0$,

$$E(exp(-\phi W)) = 1 + \sum_{\mathbf{x}\in\mathbf{R}_m^s} \tilde{e}_{\mathbf{x}} \frac{\phi}{\phi - \bar{\mu}_{\mathbf{x}}}, \tag{4.44}$$

where $\tilde{\mu}_{\mathbf{x}} = \mu_{\mathbf{x}}(1,0)$, and the $c(s)$-dimensional column vector $\tilde{\mathbf{e}}$ with elements $\tilde{e}_{\mathbf{x}}$ is given by $\tilde{\mathbf{e}} = \mathbf{E}\mathbf{K}(0)^{-1}\bar{\mathcal{X}}(0)$, with $\mathbf{E} = \mathbf{E}(1,0)$, and $\bar{\mathcal{X}}(0)$ is given in section 4.4. This result coincides with (4.3) in de Smit [21].

4.6 The virtual waiting time

Let N_t be the number of customers arriving during $[0,t]$ and let W_t^* be the virtual waiting time. Then

$$W_t^* = [W_{N_t} + V_{N_t} + T_{N_t} - t]^+ . \tag{4.45}$$

Since

$$\begin{aligned} &E(exp(-\phi(W_n + V_n + T_n - t))1(N_t = n)|C_0 = \gamma) \\ &= \int_0^t exp(\phi(t-u))(1 - F(t-u))d_u E(exp(-\phi(W_n + V_n))1(T_n \leq u)|C_0 = \gamma), \end{aligned}$$

then by using (4.26) we have for $\gamma \geq s$, $\mathbf{X}_1 \in \mathbf{R}_m^{s-1}$, and $Re(\eta) > Re(\phi) \geq 0$,

$$\begin{aligned} &\int_0^\infty exp(-\eta t)E(exp(-\phi(W_{N_t} + V_{N_t} + T_{N_t} - t))|C_0 = \gamma)dt \\ &= \frac{1 - A(\eta - \phi)}{\eta - \phi} \sum_{n=1}^\infty E(exp(-\phi(W_n + V_n) - \eta T_n)|C_0 = \gamma) \\ &= \frac{1 - A(\eta - \phi)}{\eta - \phi} \sum_{k=1}^{s-1} \sum_{\mathbf{y}\in\mathbf{R}_m^k} \sum_{i=1}^m p_i Z(1, \eta, \gamma; \mathbf{y} - \mathbf{1}_i) \\ &\quad + \frac{1 - A(\eta - \phi)}{\eta - \phi} \sum_{\mathbf{x}\in\mathbf{R}_m^s} \left(\frac{V_1(1,\eta,\gamma;\mathbf{x})}{\phi - \mu_{\mathbf{x}}(1,\eta)} - \sum_{i=1}^a \frac{a_i(\mathbf{x})V_2(1,\eta;\mathbf{x})}{(\phi + w)^{a-i+1}} \right) \\ &\quad \cdot \left(\frac{1}{A(\eta - \mu_{\mathbf{x}}(1,\eta))} \sum_{\mathbf{y}\in\mathbf{R}_m^{s-1}} \mathbf{B}_{\mathbf{y},\mathbf{x}} \right) . \end{aligned} \tag{4.46}$$

Using the identity

$$exp(-\phi x^+) = \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \frac{\phi}{\phi - \xi} exp(-\xi x), \quad Re(\phi) > Re(\xi) > 0,$$

which holds for any real x, and Theorem 16.20 in Apostol[5], we find for $\gamma \geq s$, $\mathbf{X}_1 \in \mathbf{R}_m^{s-1}$, and $Re(\eta) > 0, Re(\xi) \geq 0$,

$$\begin{aligned}\int_0^\infty exp(-\eta t)E(exp(-\phi W_t^*)|C_0 = \gamma)dt =& \frac{1}{\eta} + \sum_{\mathbf{x}\in\mathbf{R}_m^s} n_{\mathbf{x}}(\eta,\gamma)\frac{\phi}{\phi - \mu_{\mathbf{x}}(1,\eta)} \\ &+ \sum_{i=1}^{a} o_i(\eta)\frac{\phi}{(\phi+w)^{2(a-i)}},\end{aligned} \tag{4.47}$$

where

$$\begin{aligned} n_{\mathbf{x}}(\eta) &= \frac{1 - A(\eta - \mu_{\mathbf{x}}(1,\eta))}{(\eta - \mu_{\mathbf{x}}(1,\eta))A(\eta - \mu_{\mathbf{x}}(1,\eta))}\frac{V_1(1,\eta;\mathbf{x})}{\mu_{\mathbf{x}}(1,\eta)}\sum_{\mathbf{y}\in\mathbf{R}_m^{s-1}} \mathbf{B}_{\mathbf{y},\mathbf{x}} \\ o_i(\eta) &= \frac{\prod_{j=1}^{a-i} -2(j-1)}{(a-i)!}\sum_{\mathbf{x}\in\mathbf{R}_m^s}\frac{a_i(x)V_2(1,\eta;\mathbf{x})}{A(\eta - \mu_{\mathbf{x}}(1,\eta))}\sum_{\mathbf{y}\in\mathbf{R}_m^{s-1}}\mathbf{B}_{\mathbf{y},\mathbf{x}} \\ &\quad\cdot\frac{\partial^{a-i}}{\partial\xi^{a-i}}\frac{(1 - A(\eta-\xi))}{\xi(\eta-\xi)}\bigg|_{\xi=-w}. \end{aligned}$$

Furthermore, for $\gamma < s$ or $(\gamma \geq s, \mathbf{X}_1 \notin \mathbf{R}_m^{s-1})$, we have

$$\int_0^\infty exp(-\eta t)E(exp(-\phi W_t^*)|C_0 = \gamma)dt = \frac{1}{\eta} + \sum_{\mathbf{x}\in\mathbf{R}_m^s} \tilde{n}_{\mathbf{x}}(\eta,\gamma)\frac{\phi}{\phi - \mu_{\mathbf{x}}(1,\eta)}, \tag{4.48}$$

where

$$\begin{aligned} \tilde{n}_{\mathbf{x}}(\eta,\gamma) &= \frac{1 - A(\eta - \mu_{\mathbf{x}}(1,\eta))}{(\eta - \mu_{\mathbf{x}}(1,\eta))A(\eta - \mu_{\mathbf{x}}(1,\eta))}\frac{V(1,\eta,\gamma;\mathbf{x})}{\mu_{\mathbf{x}}(1,\eta)}\sum_{\mathbf{y}\in\mathbf{R}_m^{s-1}} \mathbf{B}_{\mathbf{y},\mathbf{x}} \\ &= \frac{1 - A(\eta - \mu_{\mathbf{x}}(1,\eta))}{(\eta - \mu_{\mathbf{x}}(1,\eta))A(\eta - \mu_{\mathbf{x}}(1,\eta))}e_{\mathbf{x}}(1,\eta,\gamma), \end{aligned}$$

and the function $e_{\mathbf{x}}(r,\eta,\gamma)$ is defined in (4.40).

For $\rho < 1$, let W^* be the virtual waiting time in steady state. Then by applying Abel's theorem to (4.47) we obtain for $Re(\phi) \geq 0$,

$$E(exp(-\phi W^*)) = 1 + \sum_{\mathbf{x}\in\mathbf{R}_m^s} \hat{n}_{\mathbf{x}}\frac{\phi}{\phi - \tilde{\mu}_x}, \tag{4.49}$$

with

$$\hat{n}_{\mathbf{x}} = \frac{(A(-\tilde{\mu}_{\mathbf{x}}) - 1)}{\alpha\tilde{\mu}_{\mathbf{x}}A(-\tilde{\mu}_{\mathbf{x}})}\frac{\hat{V}(\mathbf{x})}{\tilde{\mu}_{\mathbf{x}}}\sum_{\mathbf{y}\in\mathbf{R}_m^{s-1}} B_{\mathbf{y},\mathbf{x}} = \frac{(A(-\tilde{\mu}_{\mathbf{x}}) - 1)}{\alpha\tilde{\mu}_{\mathbf{x}}A(-\tilde{\mu}_{\mathbf{x}})}\tilde{e}_{\mathbf{x}},$$

where the vector $\tilde{\mathbf{V}}$ with components $\tilde{\mathbf{V}}(\mathbf{x}), \mathbf{x} \in \mathbf{R}_m^s$ is given by

$$\tilde{\mathbf{V}} = \mathbf{C}\mathbf{K}^{-1}(1,0,0)\bar{\mathcal{X}}(0).$$

This result coincides with (3.4) in de Smit [23].

4.7 The queue length at arrival epochs

Let Q_n be the number of waiting customers just before the arrival epoch T_n. To analyze its probability distribution, we first determine the events related to the events

$$\{Q_n \le j, n = 1, 2, \cdots\}.$$

The events for $\gamma < s$ are different from those for $\gamma \ge s$. Hence, we split up the study into two different cases.

4.7.1 The queue length at arrival epochs for $\gamma < s$

For $\gamma < s$, let $\check{c} = s - \gamma$. Then up to $T_{\check{c}}$ there is no queue in the system, and the random variable Q_n fulfills the following expressions,

$$\begin{aligned} \{Q_n \le j\} &= \Omega, && 1 \le n \le \check{c} + j + 1, \\ \{Q_{\check{c}+j+1+n} \le j\} &= \{T_{\check{c}+n} + W_{\check{c}+n} < T_{\bar{c}+j+1+n}\}, && n = 1, 2, \cdots, \end{aligned}$$

where Ω is the sure event. Since

$$E(q^{Q_n}) = (1-q) \sum_{j=0}^{\infty} q^j P(Q_n \le j), \tag{4.50}$$

we have for $\gamma < s$ and $|r| < 1, |q| < 1$,

$$\begin{aligned} &\sum_{n=1}^{\infty} r^n E(q^{Q_n} | C_0 = \gamma) \\ =&\sum_{j=0}^{\infty} \sum_{n=1}^{\check{c}+j+1} q^j (1-q) r^n \\ &+ \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} q^j (1-q) r^{\check{c}+j+1+n} P(T_{\check{c}+n} + W_{\check{c}+n} < T_{\check{c}+j+1+n}). \end{aligned} \tag{4.51}$$

Since $G(x)$ is continuous, then by using the identity

$$1(x < 0) + \frac{1}{2} 1(x = 0) = \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} exp(-\xi x), \tag{4.52}$$

we rewrite (4.51) as

$$
\begin{aligned}
&\sum_{n=1}^{\infty} r^n E(q^{Q_n}|C_0=\gamma) \\
=&\frac{1-q}{1-r}\left[\frac{r}{1-q}-\frac{r^{\check{c}+2}}{1-qr}\right] \\
&+\sum_{j=0}^{\infty}\sum_{n=1}^{\infty} q^j(1-q)r^{\check{c}+j+1+n}\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}A^{j+1}(-\xi)E(exp(-\xi W_{\check{c}+n})|C_0=\gamma) \\
=&\frac{1-q}{1-r}\left[\frac{r}{1-q}-\frac{r^{\check{c}+2}}{1-qr}\right] \\
&+\sum_{j=0}^{\infty} q^j(1-q)r^{j+1}\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}A^{j+1}(-\xi)\sum_{n=1}^{\infty} r^n E(exp(-\xi W_n)|C_0=\gamma) \\
&-\sum_{j=0}^{\infty} q^j(1-q)r^{j+1}\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}A^{j+1}(-\xi)\sum_{n=1}^{\check{c}} r^n E(exp(-\xi W_n)|C_0=\gamma).
\end{aligned}
\tag{4.53}
$$

Since $\gamma < s$ then $W_n = 0, \quad n = 1, 2, \cdots, \check{c}$. As a consequence, the last integral in (4.53) is equal to $\sum_{n=1}^{\check{c}} r^n$. The other integral can be analyzed by first interchanging the order of the first summation and the integration so that the summation can be replaced by a simpler term, and then expressing the term $\sum_{n=1}^{\infty} r^n E(exp(-\xi W_n)|C_0 = \gamma)$ in terms of (4.43). It yields for $\gamma < s$ and $|r| < 1, |q| < 1$,

$$
\begin{aligned}
\sum_{n=1}^{\infty} r^n E(q^{Q_n}|C_0=\gamma) =&\frac{1-q}{1-r}\left[\frac{r}{1-q}-\frac{r^{\check{c}+2}}{1-qr}\right]+\frac{r^2(1-q)}{(1-r)(1-qr)} \\
&+\sum_{\mathbf{x}\in\mathbf{R}_m^s}\frac{(1-q)rA(-\mu_{\mathbf{x}}(r,0))e_{\mathbf{x}}(r,0,\gamma)}{1-qrA(-\mu_{\mathbf{x}}(r,0))} \\
&-\frac{r^2(1-r^{\check{c}})(1-q)}{(1-r)(1-qr)} \\
=&\frac{r}{1-r}+\sum_{\mathbf{x}\in\mathbf{R}_m^s}\frac{(1-q)rA(-\mu_{\mathbf{x}}(r,0))e_{\mathbf{x}}(r,0,\gamma)}{1-qrA(-\mu_{\mathbf{x}}(r,0))},
\end{aligned}
\tag{4.54}
$$

where $e_{\mathbf{x}}(r, 0, \gamma)$ is defined in (4.40).

If $\rho < 1$ let Q be distributed according to the stationary queue-length distribution. Using Abel's limit theorem it follows from (4.54) that for $|q| \leq 1$,

$$
E(q^Q) = 1 + \sum_{\mathbf{x}\in\mathbf{R}_m^s} \tilde{e}_{\mathbf{x}}\frac{(1-q)A(-\tilde{\mu}_{\mathbf{x}})}{1-qA(-\tilde{\mu}_x)}, \tag{4.55}
$$

where $\tilde{e}_{\mathbf{x}}$ and $\tilde{\mu}_{\mathbf{x}}$ are defined as in (4.44). Equation (4.55) coincides with (4.5) in de Smit [21].

The generating function of the kth moment of Q_n can also be derived from (4.54). For $\gamma < s$, the generating function of the first moment is given in the following equation,

$$\sum_{n=1}^{\infty} r^n E(Q_n | C_0 = \gamma) = - \sum_{\mathbf{x} \in \mathbf{R}_m^s} \frac{rA(-\mu_{\mathbf{x}}(r,0)) e_{\mathbf{x}}(r,0,\gamma)}{1 - rA(-\mu_{\mathbf{x}}(r,0))}. \tag{4.56}$$

If $\rho < 1$ then it follows from (4.55) in the usual way that

$$E(Q) = - \sum_{\mathbf{x} \in \mathbf{R}_m^s} \tilde{e}_{\mathbf{x}} \frac{A(-\tilde{\mu}_{\mathbf{x}})}{1 - A(-\tilde{\mu}_{\mathbf{x}})}, \tag{4.57}$$

where $\tilde{e}_{\mathbf{x}}$ and $\tilde{\mu}_{\mathbf{x}}$ are defined as in (4.44).

4.7.2 The queue length at arrival epochs for $\gamma \geq s$

For $\gamma \geq s$, at time T_1 we already find a queue, since at this epoch all servers can serve only the first s special customers, and the rest is waiting for service. Let $\tilde{W}_n$ be the waiting time of the nth special customer. The random variable Q_n satisfies the relations

$$\{Q_1 \leq j\} = \begin{cases} \text{impossible event} & \text{, for} \quad j = 0, 1, \cdots, C_0 - s - 1, \\ \Omega & \text{, for} \quad j = C_0 - s, C_0 - s + 1, \cdots, \end{cases}$$

$$\{Q_2 \leq 0\} = \{T_{n-j-1} + W_{n-j-1} < T_n\},$$

for $j = 1, 2, \cdots, C_0$

$$\{Q_n \leq j\} = \begin{cases} \{\tilde{W}_{C_0+n-j-1} < T_n\} & \text{, for} \quad n = 2, 3, \cdots, j+1, \\ \{T_{n-j-1} + W_{n-j-1} < T_n\} & \text{, for} \quad n = j+2, j+3, \cdots, \end{cases}$$

and for $j = C_0 + 1, C_0 + 2, \cdots,$

$$\{Q_n \leq j\} = \begin{cases} \Omega & \text{, for} \quad n = 2, 3, \cdots, j - C_0 + 1, \\ \{\tilde{W}_{C_0+n-j-1} < T_n\} & \text{, for} \quad n = j - C_0 + 2, \cdots, j+1, \\ \{T_{n-j-1} + W_{n-j-1} < T_n\} & \text{, for} \quad n = j+2, j+3, \cdots. \end{cases}$$

Since $G(x)$ and $I(x)$ are continuous, it follows by using the identity (4.52), that for $\gamma \geq s$ and $|r| < 1, |q| < 1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} r^n E(q^{Q_n}|C_0 = \gamma) \\
&= (1-q)r \sum_{j=\gamma-s}^{\infty} q^j + (1-q)\sum_{j=0}^{\gamma} q^j \sum_{n=2}^{\infty} r^n P(Q_n \leq j) \\
&\quad + (1-q) \sum_{j=\gamma+1}^{\infty} q^j \sum_{n=2}^{\infty} r^n P(Q_n \leq j)
\end{aligned} \tag{4.58}$$

By exploring the events $\{Q_n \leq j\}$ as we did above, we then obtain for $\gamma \geq s$ and $|r| < 1$, $|q| < 1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} r^n E(q^{Q_n}|C_0 = \gamma) \\
&= rq^{\gamma-s} + (1-q)\sum_{j=1}^{\gamma} q^j \sum_{n=2}^{j+1} r^n P(\tilde{W}_{\gamma+n-j-1} < T_n) \\
&\quad + (1-q)\sum_{j=0}^{\gamma} q^j \sum_{n=j+2}^{\infty} r^n P(T_{n-j-1} + W_{n-j-1} < T_n) \\
&\quad + (1-q)\left[\sum_{j=\gamma+1}^{\infty} q^j \sum_{n=2}^{j-\gamma+1} r^n + \sum_{j=\gamma+1}^{\infty} q^j \sum_{n=j-\gamma+2}^{j+1} r^n P(\tilde{W}_{\gamma+n-j-1} < T_n)\right] \\
&\quad + (1-q) \sum_{j=\gamma+1}^{\infty} q^j \sum_{n=j+2}^{\infty} r^n P(T_{n-j-1} + W_{n-j-1} < T_n) \qquad (4.59) \\
&= rq^{\gamma-s} + (1-q)r^2 \sum_{j=\gamma+1}^{\infty} q^j \frac{(1-r^{j-\gamma})}{(1-r)} \\
&\quad + (1-q)r\sum_{j=0}^{\infty}(qr)^j \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \sum_{n=1}^{\infty} r^n A^{j+1}(-\xi) E(exp(-\xi W_n)|C_0 = \gamma) \\
&\quad + (1-q)\sum_{j=1}^{\gamma} q^j \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \sum_{n=2}^{j+1} r^n A^{n-1}(-\xi) E(exp(-\xi \tilde{W}_{\gamma+n-j-1})) \\
&\quad + (1-q)r^{1-\gamma} \sum_{j=\gamma+1}^{\infty} (qr)^j \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \sum_{n=1}^{\gamma} r^n A^{n-\gamma+j}(-\xi) E(exp(-\xi \tilde{W}_n)).
\end{aligned}$$

Since we assume that all the special customers have a common exponential service time with rate w and are served first come first serve, then the departure process of the special customers is a Poisson process with rate sw. It follows that for $n = s+1, s+2, \cdots, C_0$,

$$E\left[e^{-\xi \tilde{W}_n}\right] = \left(\frac{sw}{sw+\xi}\right)^{n-s}.$$

If we substitute this into (4.59) and if we express $\sum_{n=1}^{\infty} r^n E(exp(-\xi W_n)|\, C_0 = \gamma)$ in terms of (4.43), by applying a contour integration we then obtain for $\gamma \geq s$ and $|r| < 1, |q| < 1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} r^n E(q^{Q_n}|C_0 = \gamma) \\
&= rq^{\gamma-s} + (1-q)r^2 \sum_{j=\gamma+1}^{\infty} q^j \frac{(1-r^{j-\gamma})}{(1-r)} \\
&\quad + \frac{r^2(1-q)}{(1-r)(1-qr)} + \sum_{\mathbf{x}\in\mathbf{R}_m^s} \frac{(1-q)rA(-\mu_{\mathbf{x}}(r,0))e_{\mathbf{x}}(r,0,\gamma)}{1-qrA(-\mu_{\mathbf{x}}(r,0))} \\
&\quad + (1-q)\sum_{j=1}^{\gamma+1-s} q^j (sw) d(j-1,0) \\
&\quad + (1-q)\sum_{j=1}^{\gamma+1-s} q^j (sw)^{\gamma+1-j-s}(\gamma-s-j-1)! d_1(j) \\
&\quad + (1-q)\frac{r^{2-\gamma}}{(1-r)}\left[\frac{(qr)^{\gamma-s+2}-(qr)^{\gamma+1}}{(1-qr)} - \frac{r^{s-\gamma}(qr^2)^{\gamma-s+2}-(qr^2)^{\gamma+1}}{(1-qr^2)}\right] \\
&\quad + (1-q)r^{s-\gamma}\sum_{j=\gamma+2-s}^{\gamma} (qr)^j \left[(sw)^{s+1-\gamma} d(\gamma-s-1,0) + (sw)^{s+1}(\gamma-1)! d_2(j)\right] \\
&\quad + (1-q)r^{2-\gamma}\frac{(1-r^{s+1})}{(1-r)}\frac{(qr)^{\gamma+1}}{(1-qr)} + (1-q)r^{2-\gamma+s}\frac{(1-r^{\gamma-s-1})}{(1-r)}\frac{(qr)^{\gamma+1}}{(1-qr)},
\end{aligned} \tag{4.60}$$

where

$$d(j,\xi) = \sum_{n=0}^{j-1} (sw)^{j-1-n}(rA(-\xi)sw)^n, \tag{4.61}$$

$$d_1(j) = \frac{\partial^{\gamma-s-j-1}(A(-\xi)d(j-1,\xi)/\xi)}{\partial\xi^{\gamma-s-j-1}}\Big|_{\xi=-sw}, \tag{4.62}$$

and

$$d_2(j) = \frac{\partial^{\gamma-1}(A^{s+1-\gamma+j}(-\xi)d(\gamma-s+1,\xi)/\xi)}{\partial\xi^{\gamma-1}}\Big|_{\xi=-sw}. \tag{4.63}$$

The generating function of the kth moment of Q_n can also be derived from (4.60). For $\gamma \geq s$, the generating function of the first moment is given in the following equation. For

$|r| < 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} r^n E(Q_n|C_0=\gamma) = r(\gamma-s) - r^2 \sum_{j=\gamma+1}^{\infty} \frac{(1-r^{j-\gamma})}{1-r} - \frac{r^2}{(1-r)^2} \\
+ \sum_{\mathbf{x}\in\mathbf{R}_m^s} \frac{rA(-\mu_{\mathbf{x}}(r,0))e_{\mathbf{x}}(r,0,\gamma)}{1-rA(-\mu_{\mathbf{x}}(r,0))} - (sw)\sum_{j=1}^{\gamma+1-s} d(j-1,0) \\
- \sum_{j=1}^{\gamma+1-s} (sw)^{\gamma+1-j-s}(\gamma-s-j-1)!d_1(j) \\
- \frac{r^{4-s}-r^3}{(1-r)^2} + \frac{r^{6-s}-r^{4+\gamma}}{(1-r)(1-r^2)} \\
- r^{s-\gamma} \sum_{j=\gamma+2-s}^{\gamma} (r)^j (sw)^{s+1-\gamma} d(\gamma-s-1,0) \\
- r^{s-\gamma} \sum_{j=\gamma+2-s}^{\gamma} (r)^j (sw)^{s+1} (\gamma-1)! d_2(j) \\
- \frac{r^3(1-r^{s+1}) + r^{3+s}(1-r^{\gamma-s-1})}{(1-r)^2}.
\end{aligned} \tag{4.64}$$

If $\rho < 1$, then from (4.64) in the usual way we can obtain the expression for $E(Q)$, which coincides with (4.57).

4.8 The total number of customers at arrival epochs

Let C_n be the number of customers in the system just before the arrival epoch T_n. We have that

$$\begin{aligned}
P(C_n=k|C_0=\gamma) &= \sum_{\mathbf{x}\in\mathbf{R}_m^k} P(\mathbf{X}_n=\mathbf{x}|C_0=\gamma), \qquad k=0,1,\cdots,s-2 \\
P(C_n=s-1|C_0=\gamma) &= \sum_{\mathbf{x}\in\mathbf{R}_m^{s-1}} \lim_{\phi\to\infty} \mathcal{Z}_n(0,\phi,\gamma;\mathbf{x}), \\
P(C_n=s|C_0=\gamma) &= P(Q_n=0|C_0=\gamma) - \sum_{k=0}^{s-1} P(C_n=k|C_0=\gamma), \\
P(C_n=k|C_0=\gamma) &= P(Q_n=k-s|C_0=\gamma), \qquad k=s+1,s+2,\cdots;
\end{aligned}$$

so that we have for $|r| < 1, |q| \leq 1$,

$$\sum_{n=1}^{\infty} r^n E(q^{C_n}|C_0 = \gamma) = \sum_{k=0}^{s-2}(q^k - q^s) \sum_{\mathbf{x}\in\mathbf{R}_m^k} Z(r, 0, \gamma; \mathbf{x}) + q^{s-1}(1-q) \lim_{\phi\to\infty} \sum_{\mathbf{x}\in\mathbf{R}_m^{s-1}} \mathcal{Z}(r, 0, \phi, \gamma; \mathbf{x}) + q^s \sum_{n=1}^{\infty} r^n E(q^{Q_n}|C_0 = \gamma). \tag{4.65}$$

4.8.1 The total number of customers at arrival epochs for $\gamma < s$

Let $\mathbf{1}$ be the $c(s-1)$-dimensional row vector with all components equal to 1. From Theorem 4.13 and Lemma 4.3.1 it follows that

$$\lim_{\phi\to\infty} \sum_{\mathbf{x}\in\mathbf{R}_m^{s-1}} \mathcal{Z}(r, 0, \phi, \gamma; \mathbf{x}) = \mathbf{1K}(r, 0, 0)^{-1}\mathcal{Z}(r, 0, 0, \gamma).$$

If we substitute this into (4.65), then by using (4.54) we obtain for $C_0 = \gamma < s$, $|r| < 1, |q| < 1$,

$$\begin{aligned} &\sum_{n=1}^{\infty} r^n E(q^{C_n}|C_0 = \gamma) \\ &= \sum_{k=0}^{s-2}(q^k - q^s) \sum_{\mathbf{x}\in\mathbf{R}_m^k} Z(r, 0, \gamma; \mathbf{x}) + q^{s-1}(1-q)\mathbf{1K}(r, 0, 0)^{-1}\bar{\mathcal{Z}}(r, 0, 0, \gamma) \\ &\quad + q^s \frac{r}{1-r} + q^s \sum_{\mathbf{x}\in\mathbf{R}_m^s} \frac{(1-q)rA(-\mu_{\mathbf{x}}(r, 0))e_{\mathbf{x}}(r, 0, \gamma)}{1 - qrA(-\mu_{\mathbf{x}}(r, 0)}. \end{aligned} \tag{4.66}$$

For $\rho < 1$, C_n weakly converges to a random variable C. Then for $|q| < 1$,

$$E(q^C) = \sum_{k=0}^{s-2}(q^k - q^s) \sum_{\mathbf{x}\in R_m^k} \bar{Z}(\mathbf{x}) + q^{s-1}(1-q)\mathbf{1K}(0)^{-1}\bar{\mathcal{X}}(0) + q^s \left[1 + \sum_{\mathbf{x}\in\mathbf{R}_m^s} \tilde{e}_{\mathbf{x}} \frac{(1-q)A(-\bar{\mu}_{\mathbf{x}})}{1 - qA(-\bar{\mu}_{\mathbf{x}})}\right], \tag{4.67}$$

where $\tilde{e}_{\mathbf{x}}$ and $\tilde{\mu}_{\mathbf{x}}$ are defined as in (4.44). The equation (4.67) coincides with (4.9) in [21].

The generating function of the kth moment of C_n can be derived from (4.66). For $\gamma < s$, the generating function of the first moment is given by the following expression.

For $|r| < 1$,

$$\begin{aligned}\sum_{n=1}^{\infty} r^n E(C_n|C_0=\gamma) = &\sum_{k=0}^{s-2}(k-s)\sum_{\mathbf{x}\in\mathbf{R}_m^k} Z(r;\mathbf{x}) - \mathbf{1}K^{-1}(r,0,0)Z(r,0,0,\gamma) \\ &+ s\frac{r}{1-r} - \sum_{\mathbf{x}\in\mathbf{R}_m^s}\frac{(rA(-\mu_{\mathbf{x}}(r,0))e_{\mathbf{x}}(r,0,\gamma)}{1-rA(-\mu_{\mathbf{x}}(r,0))}.\end{aligned} \tag{4.68}$$

If $\rho < 1$ then we have

$$\begin{aligned}E(C) = &\sum_{k=0}^{s-2}(k-s)\sum_{\mathbf{x}\in\mathbf{R}_m^k} Z(\mathbf{x}) - \mathbf{1}K^{-1}(0)\bar{\mathcal{X}}(0) + s \\ &- \sum_{\mathbf{x}\in R_m^s}\tilde{e}_{\mathbf{x}}\frac{A(-\tilde{\mu}_x)}{1-A(-\tilde{\mu}_x)},\end{aligned} \tag{4.69}$$

where $\tilde{e}_x$ and $\tilde{\mu}_{\mathbf{x}}$ are defined as in (4.44).

4.8.2 The total number of customers at arrival epochs for $\gamma \geq s$

Let $\mathbf{1}$ be the $c(s-1)$-dimensional row vector with all components equal to 1. From Theorem 4.13 and Lemma 4.3.1 it follows that

$$\lim_{\phi\to\infty}\sum_{\mathbf{x}\in\mathbf{R}_m^{s-1}}\mathcal{Z}(r,0,\phi,\gamma;\mathbf{x}) = \mathbf{1}\mathbf{K}(r,0,0)^{-1}\mathcal{Z}(r,0,0,\gamma) - \mathbf{1}\bar{\mathcal{Z}}_1^+(r,0,0,\gamma).$$

If we substitute this into (4.65), then by using (4.54) we obtain for $\gamma \geq s, |r| < 1, |q| < 1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} r^n E(q^{C_n}|C_0 = \gamma) \\
&= \sum_{k=0}^{s-2} (q^k - q^s) \sum_{\mathbf{x}\in\mathbf{R}_m^k} Z(r,0,\gamma;\mathbf{x}) + q^{s-1}(1-q)\mathbf{1}\mathbf{K}(r,0,0)^{-1}\bar{\mathcal{Z}}(r,0,0,\gamma) \\
&+ q^{s-1}(1-q)\mathbf{1}\bar{\mathcal{Z}}_1^{+}(r,0,0,\gamma) + rq^{\gamma} + (1-q)q^s r^2 \sum_{j=\gamma+1}^{\infty} q^j \frac{(1-r^{j-\gamma})}{1-r} \\
&\quad + \frac{r^2(1-q)q^s}{(1-r)(1-qr)} + q^s \sum_{\mathbf{x}\in\mathbf{R}_m^s} \frac{(1-q)rA(-\mu_{\mathbf{x}}(r,0))e_{\mathbf{x}}(r,0,\gamma)}{1-qrA(-\mu_{\mathbf{x}}(r,0))} \\
&\quad + (1-q)q^s \sum_{j=1}^{\gamma+1-s} q^j (sw) d(j-1, 0) \\
&\quad + (1-q)q^s \sum_{j=1}^{\gamma+1-s} q^j (sw)^{\gamma+1-j-s}(\gamma-s-j-1)! d_1(j) \\
&\quad + (1-q)q^s \frac{r^{2-\gamma}}{(1-r)} \left[\frac{(qr)^{\gamma-s+2} - (qr)^{\gamma+1}}{1-qr} - \frac{r^{s-\gamma}(qr^2)^{\gamma-s+2} - (qr^2)^{\gamma+1}}{1-qr^2} \right] \\
&\quad + (1-q)q^s r^{s-\gamma} \sum_{j=\gamma+2-s}^{\gamma} (qr)^j (sw)^{s+1-\gamma} d(\gamma-s-1, 0) \\
&\quad + (1-q)q^s r^{s-\gamma} \sum_{j=\gamma+2-s}^{\gamma} (qr)^j (sw)^{s+1}(\gamma-1)! d_2(j) \\
&\quad + (1-q)q^s r^{2-\gamma} \frac{(1-r^{s+1})}{(1-r)} \frac{(qr)^{\gamma+1}}{(1-qr)} + (1-q)r^{2-\gamma+s} \frac{(1-r^{\gamma-s-1})}{(1-r)} \frac{(qr)^{\gamma+1}}{(1-qr)},
\end{aligned} \tag{4.70}$$

where the functions d, d_1, and d_2 are defined in (4.61) - (4.63).

The generating function of the kth moment of C_n can be derived from (4.70). For $\gamma \geq s$, the generating function of the first moment is given by the following expression.

For $|r| < 1$,

$$
\begin{aligned}
\sum_{n=1}^{\infty} r^n E(C_n|C_0 = \gamma) = &\sum_{k=0}^{s-2}(k-s) \sum_{\mathbf{x}\in\mathbf{R}_m^k} Z(r;\mathbf{x}) - \mathbf{1}K^{-1}(r,0,0)Z(r,0,0,\gamma) \\
&- \mathbf{1}\bar{\mathcal{Z}}_1^{+}(r,0,0,\gamma) + \gamma r - r^2 \sum_{j=\gamma+1}^{\infty} \frac{(1-r^{j-\gamma})}{1-r} \\
&- \frac{r^2}{(1-r)^2} - \sum_{\mathbf{x}\in\mathbf{R}_m^s} \frac{(rA(-\mu_{\mathbf{x}}(r,0))e_{\mathbf{x}}(r,0,\gamma)}{(1-rA(-\mu_{\mathbf{x}}(r,0))} \\
&- \sum_{j=1}^{\gamma+1-s} (sw)d(j-1,0) + (sw)^{\gamma+1-j-s}(\gamma-s-j-1)!d_1(j) \\
&- \frac{r^{4-s}-r^3}{(1-r)^2} + \frac{r^{6-s}-r^{4+\gamma}}{(1-r)(1-r^2)} \\
&- r^{s-\gamma} \sum_{j=\gamma+2-s}^{\gamma} (r)^j (sw)^{s+1-\gamma} d(\gamma-s-1,0) \\
&- r^{s-\gamma} \sum_{j=\gamma+2-s}^{\gamma} (r)^j (sw)^{s+1}(\gamma-1)!d_2(j) \\
&- \frac{r^3(1-r^{s+1}) + r^{3+s}(1-r^{\gamma-s-1})}{(1-r)^2}.
\end{aligned}
\tag{4.71}
$$

If $\rho < 1$ then from (4.71) in the usual way we obtain the expression for $E(C)$, which coincides with (4.69).

4.9 Queue length in continuous time

Let Q_t^* be the queue length at time t. Its sample functions are considered to be left continuous. By conditioning on the number of arrivals up to time t we have for $j > \gamma$ and

$t > 0$,

$$\begin{aligned}
&P(Q_t^* \leq j) \\
&= P(N_t \leq j-\gamma) + \sum_{n=1}^{\infty} P(Q_t^* \leq j, N_t = n+j-\gamma) \\
&= P(T_{j-\gamma+1} > t) + \sum_{n=1}^{\gamma} P(\tilde{W}_n < t, T_{n+j-\gamma} \leq t < T_{n+j-\gamma+1}) \\
&\quad + \sum_{n=1}^{\infty} P(T_n + W_n < t, T_{n+j} \leq t < T_{n+j+1}) \\
&= 1 - P(T_{j-\gamma+1} \leq t) + \int_0^t \{1-F(t-u)\} d_u \sum_{n=1}^{\gamma} P(\tilde{W}_n < t, T_{n+j-\gamma} \leq u) \\
&\quad + \int_0^t \{1-F(t-u)\} d_u \sum_{n=1}^{\infty} P(T_n + W_n < t, T_{n+j} \leq u).
\end{aligned} \tag{4.72}$$

Since $G(x)$ is absolutely continuous, and we assume that the function $F(x)$ is continuous, (4.72) becomes

$$\begin{aligned}
P(Q_t^* \leq j) = 1 &- P(T_{j-\gamma+1} \leq t) \\
&+ \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \int_0^t \{1-F(t-u)\} \\
&\quad .d_u \sum_{n=1}^{\gamma} E(exp(-\xi(\tilde{W}_n - t))\mathbf{1}(T_{n+j-\gamma} \leq u)) \\
&+ \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \int_0^t \{1-F(t-u)\} \\
&\quad .d_u \sum_{n=1}^{\infty} E(exp(-\xi(T_n + W_n - t))1(T_{n+j} \leq u)).
\end{aligned} \tag{4.73}$$

For $j = 0$, we obtain

$$P(Q_t^* = 0) = \sum_{n=1}^{\infty} P(Q_t^* = 0, N_t = n) = \sum_{n=1}^{\infty} P(T_n + W_n < t, T_n \leq t < T_{n+1}),$$

and for $0 < j \leq \gamma$,

$$
\begin{aligned}
&P(Q_t^* \le j) \\
&= \sum_{n=1}^{\infty} P(Q_t^* \le j, N_t = n) \\
&= \sum_{n=1}^{j} P(\tilde{W}_{\gamma-j+n} < t, T_n \le t < T_{n+1}) \\
&\quad + \sum_{n=j+1}^{\infty} P(T_{n-j} + W_{n-j} < t, T_n \le t < T_{n+1}) \\
&= \int_0^t \{1 - F(t-u)\} d_u \sum_{n=\gamma-j+1}^{\gamma} P(\tilde{W}_n < t, T_{n-\gamma+j} \le u) \\
&\quad + \int_0^t \{1 - F(t-u)\} d_u \sum_{n=1}^{\infty} P(T_n + W_n < t, T_{n+j} \le u) \\
&= \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \int_0^t \{1 - F(t-u)\} d_u \sum_{n=\gamma-j+1}^{\gamma} E(e^{-\xi(\tilde{W}_n - t} \mathbf{1}(T_{n-\gamma+j} \le u)) \\
&\quad + \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \int_0^t \{1 - F(t-u)\} d_u \sum_{n=1}^{\infty} E(e^{-\xi(T_n + W_n - t)} 1(T_{n+j} \le u)).
\end{aligned}
\tag{4.74}
$$

Since $E(q^{Q_t^*}) = (1-q)\sum_{j=0}^{\infty} q^j P(Q_t^* \le j)$, we have for $|q| \le 1$,

$$\begin{aligned}
&E(q^{Q_t^*}|C_0=\gamma)\\
&= (1-q)\sum_{j=1}^{\gamma} q^j \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}\int_0^t \{1-F(t-u)\}\\
&\quad .d_u \sum_{n=\gamma-j+1}^{\gamma} E(e^{-\xi(\tilde{W}_n-t)}\mathbf{1}(T_{n-\gamma+j}\le u))\\
&\quad + (1-q)\sum_{j=0}^{\gamma} q^j \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}\int_0^t \{1-F(t-u)\}\\
&\quad .d_u \sum_{n=1}^{\infty} E(e^{-\xi(T_n+W_n-t)}1(T_{n+j}\le u)|C_0=\gamma)\\
&\quad + (1-q)\sum_{j=\gamma+1}^{\infty} q^j(1-P(T_{j-\gamma+1}\le t))\\
&\quad + (1-q)\sum_{j=\gamma+1}^{\infty} q^j \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}\int_0^t \{1-F(t-u)\}\\
&\quad .d_u \sum_{n=1}^{\gamma} E(e^{-\xi(\tilde{W}_n-t)}\mathbf{1}(T_{n+j-\gamma}\le u))\\
&\quad + (1-q)\sum_{j=\gamma+1}^{\infty} q^j \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}\int_0^t \{1-F(t-u)\}\\
&\quad .d_u \sum_{n=1}^{\infty} E(e^{-\xi(T_n+W_n-t)}1(T_{n+j}\le u)|C_0=\gamma).
\end{aligned} \tag{4.75}$$

4.9.1 Queue length in continuous time for $\gamma < s$

If $\gamma < s$, it is clear that $\tilde{W}_n = 0, n = 1, 2, \cdots, \gamma$. Then from (4.75)

$$\begin{aligned}
&\int_0^\infty exp(-\eta t)E(q^{Q_t^*}|C_0 = \gamma)dt \\
=&(1-q)\left[\sum_{j=1}^{\gamma} q^j \frac{1-A(\eta)}{\eta}\sum_{n=1}^{j} A^{n-1}(\eta)\right. \\
&+\sum_{j=0}^{\gamma}\frac{q^j}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}\frac{1-A(\eta-\xi)}{(\eta-\xi)}\sum_{n=1}^{\infty}E(exp(-\eta T_n-\xi W_n)|C_0=\gamma)A^j(\eta-\xi) \\
&+\sum_{j=\gamma+1}^{\infty} q^j\left(\frac{1}{\eta}-\frac{A^{j-\gamma}(\eta)}{\eta}+\frac{(1-A(\eta))}{\eta}\sum_{n=1}^{\gamma}A^{n+j-\gamma-1}(\eta)\right. \\
&\left.\left.+\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}\frac{1-A(\eta-\xi)}{(\eta-\xi)}\sum_{n=1}^{\infty}E(exp(-\eta T_n-\xi W_n)|C_0=\gamma)A^j(\eta-\xi)\right)\right].
\end{aligned} \tag{4.76}$$

From (4.33), (4.13), (4.9) and Lemma 4.3.1, we have that for $\gamma < s$,

$$\sum_{n=1}^{\infty} E(exp(-\eta T_n - \xi W_n)|C_0 = \gamma) = \frac{1}{1-A(\eta)} + \sum_{\mathbf{x}\in\mathbf{R}_m^s} e_{\mathbf{x}}(1,\eta,\gamma)\frac{\xi}{\xi-\mu_{\mathbf{x}}(1,\eta)},$$

where $e_{\mathbf{x}}(r, \eta, \gamma)$ is defined in (4.40). Substituting this into (4.76), then by applying contour integration we obtain for $\gamma < s, |q| \le 1, Re(\eta) > 0$,

$$\begin{aligned}
&\int_0^\infty exp(-\eta t)E(q^{Q_t^*}|C_0 = \gamma)dt \\
=&(1-q)\left[\sum_{j=1}^{\gamma} q^j \frac{1-A(\eta)}{\eta}\sum_{n=1}^{j} A^{n-1}(\eta) + \frac{(1-q^{\gamma+1}A^{\gamma+1}(\eta))}{\eta(1-qA(\eta))}\right. \\
&+\sum_{\mathbf{x}\in\mathbf{R}_m^s} e_{\mathbf{x}}(1,\eta)\frac{(1-A(\eta-\mu_{\mathbf{x}}(1,\eta)))\,(1-q^{\gamma+1}A^{\gamma+1}(\eta-\mu_{\mathbf{x}}(1,\eta)))}{(\eta-\mu_{\mathbf{x}}(1,\eta))(1-qA(\eta-\mu_{\mathbf{x}}(1,\eta)))} \\
&+\sum_{\mathbf{x}\in\mathbf{R}_m^s} e_{\mathbf{x}}(1,\eta)\frac{(1-A(\eta-\mu_{\mathbf{x}}(1,\eta)))\,q^{\gamma+1}A^{\gamma+1}(\eta-\mu_{\mathbf{x}}(1,\eta))}{(\eta-\mu_{\mathbf{x}}(1,\eta))(1-qA(\eta-\mu_{\mathbf{x}}(1,\eta)))} \\
&\left.+\frac{q^{\gamma+1}}{\eta(1-q)}-\frac{q^{\gamma+1}A(\eta)}{\eta(1-qA(\eta))}+\frac{q^{\gamma+1}\sum_{n=1}^{\gamma}A^n(\eta)}{\eta(1-qA(\eta))}+\frac{q^{\gamma+1}A^{\gamma+1}(\eta)}{\eta(1-qA(\eta))}\right].
\end{aligned} \tag{4.77}$$

For $\rho < 1, |q| \le 1$, let Q^* be the queue length in continuous time in steady state. We find from (4.77) in the usual way

$$E(q^{Q^*}) = 1 + \sum_{\mathbf{x}\in\mathbf{R}_m^s} \tilde{e}_{\mathbf{x}}\frac{(1-q)(1-A(-\tilde{\mu}_{\mathbf{x}}))}{-\tilde{\mu}_{\mathbf{x}}(1-qA(-\tilde{\mu}_{\mathbf{x}}))}, \tag{4.78}$$

in accordance to equation (3.9) in [23].

4.9.2 Queue length in continuous time for $\gamma \geq s$

If $\gamma \geq s$, it is clear that $\tilde{W}_n = 0$ for $n = 1, 2, \cdots, s$, and $\tilde{W}_n > 0$ for $n = s+1, \cdots, \gamma$. It follows that

$$\begin{aligned}
&\sum_{j=1}^{\gamma} q^j \int_0^t \{1 - F(t-u)\} d_u \sum_{n=\gamma-j+1}^{\gamma} P(\tilde{W}_n < t, T_{n-\gamma+j} \leq u) \\
=& \sum_{j=1}^{\gamma-s-1} q^j \int_0^t \{1 - F(t-u)\} d_u \sum_{n=\gamma-j+1}^{\gamma} P(\tilde{W}_n < t, T_{n-\gamma+j} \leq u) \\
&+ \sum_{j=\gamma-s}^{\gamma} q^j \int_0^t \{1 - F(t-u)\} d_u \sum_{n=\gamma-j+1}^{\gamma} P(T_{n-\gamma+j} \leq u),
\end{aligned} \tag{4.79}$$

and

$$\begin{aligned}
&\int_0^t \{1 - F(t-u)\} d_u \sum_{n=1}^{\gamma} P(\tilde{W}_n < t, T_{n-\gamma+j} \leq u) \\
=& \int_0^t \{1 - F(t-u)\} d_u \sum_{n=1}^{s} P(T_{n-\gamma+j} \leq u) \\
&+ \int_0^t \{1 - F(t-u)\} d_u \sum_{n=s+1}^{\gamma} P(\tilde{W}_n < t, T_{n-\gamma+j} \leq u).
\end{aligned} \tag{4.80}$$

Furthermore we have from (4.33), (4.13), (4.9) and Lemma 4.3.1, that for $\gamma \geq s$,

$$\begin{aligned}
\sum_{n=1}^{\infty} E(exp(-\eta T_n - \xi W_n) | C_0 = \gamma) =& \frac{1}{1 - A(\eta)} + \sum_{\mathbf{x} \in \mathbf{R}_m^s} e_{\mathbf{x}}(1, \eta, \gamma) \frac{\xi}{(\xi - \mu_{\mathbf{x}}(1, \eta))} \\
&+ \mathbf{1K}(1, \eta, \xi)[\bar{\mathcal{Z}}_1^+(1, \eta, \xi, \gamma) - \bar{\mathcal{Z}}_1^+(1, \eta, 0, \gamma)].
\end{aligned} \tag{4.81}$$

If we substitute (4.79), (4.80), and (4.81) into (4.75), then similar to the case $\gamma < s$, we obtain for $\gamma \geq s, |q| \leq 1, Re(\eta) > 0$,

$$
\begin{aligned}
&\int_0^\infty exp(-\eta t)E(q^{Q_t^*}|C_0=\gamma)dt \\
=&(1-q)\left[\sum_{j=\gamma-s}^{\gamma} q^j \frac{1-A(\eta)}{\eta}\sum_{n=1}^{j}A^{n-1}(\eta)\right. \\
&+\sum_{j=1}^{\gamma-s-1} q^j\left(\frac{1-A(\eta)}{\eta}\sum_{n=\gamma-j+1}^{\gamma}\left(\frac{sw}{sw+\eta}\right)^{n-s}A^{n-\gamma+j-1}(\eta)\right) \\
&+\frac{1}{\eta(1-qA(\eta))}+\frac{q^{\gamma+1}}{\eta(1-q)}-\frac{q^{\gamma+1}A(\eta)}{\eta(1-qA(\eta))}+\frac{q^{\gamma+1}\sum_{n=1}^{s}A^n(\eta)}{\eta(1-qA(\eta))} \\
&+\sum_{\mathbf{x}\in\mathbf{R}_m^s} e_{\mathbf{x}}(1,\eta)\frac{(1-A(\eta-\mu_{\mathbf{x}}(1,\eta)))\left(1-q^{\gamma+1}A^{\gamma+1}(\eta-\mu_{\mathbf{x}}(1,\eta))\right)}{(\eta-\mu_{\mathbf{x}}(1,\eta))(1-qA(\eta-\mu_{\mathbf{x}}(1,\eta)))} \\
&+\sum_{\mathbf{x}\in\mathbf{R}_m^s} e_{\mathbf{x}}(1,\eta)\frac{(1-A(\eta-\mu_{\mathbf{x}}(1,\eta)))\,q^{\gamma+1}A(\eta-\mu_{\mathbf{x}}(1,\eta))}{(\eta-\mu_{\mathbf{x}}(1,\eta))(1-qA(\eta-\mu_{\mathbf{x}}(1,\eta)))} \\
&\left.+\sum_{j=\gamma+1}^{\infty} q^j\left(\frac{1-A(\eta)}{\eta}\sum_{n=s+1}^{\gamma}\left(\frac{sw}{sw+\eta}\right)^{n-s}A^{n-\gamma+j-1}(\eta)\right)\right].
\end{aligned}
\tag{4.82}
$$

If $\rho < 1, |q| \leq 1$, then from (4.77) in the usual way we obtain the expression for $E(q^Q)$, which coincides with (4.78).

4.10 The total number of customers in continuous time

Let C_t^* be the number of customers at time t, which we consider to be a left-continuous process. Observe that

$$P(C_t^* = s-1|C_0=\gamma) = P(W_t^*=0|C_0=\gamma) - \sum_{j=0}^{s-2} P(C_t^* = l|C_0=\gamma), \tag{4.83}$$

$$P(C_t^* = s|C_0=\gamma) = P(Q_t^*=0|C_0=\gamma) - P(W_t^*=0|C_0=\gamma) \tag{4.84}$$

$$P(C_t^* = l|C_0=\gamma) = P(Q_t^*=l-s|C_0=\gamma), \quad l=s+1, s+2, \cdots. \tag{4.85}$$

To find the $P(C_t^* = l | C_0 = \gamma)$ for $l = 0, 1, \cdots, s-2$, we can use the same derivation as in section 3.3 in [23]. It yields for $\gamma < s$, $Re(\eta) > 0$, $l = 0, 1, 2, \cdots, s-2$;

$$
\begin{aligned}
&\int_0^\infty exp(-\eta t) P(C_t^* = l | C_0 = \gamma) dt \\
&= \sum_{\mathbf{x} \in \mathbf{R}_m^l} \Bigg[\sum_{k=l}^{s-1} \sum_{\mathbf{y} \in \mathbf{R}_m^k} \sum_{\nu \leq \mathbf{y} - \mathbf{x}} \binom{\mathbf{y}}{\mathbf{x}; \nu} (-1)^{\nu \mathbf{1}} \frac{1 - A(\eta + \mathbf{xb} + \nu \mathbf{b})}{\eta + \mathbf{xb} + \nu \mathbf{b}} \\
&\quad \cdot \sum_{i=1}^m p_i Z(1, \eta, \gamma; \mathbf{y} - \mathbf{1}_i) - \sum_{\mathbf{y} \in \mathbf{R}_m^{s-1}} \sum_{\nu \leq \mathbf{y} - \mathbf{x}} \binom{\mathbf{y}}{\mathbf{x}; \nu} (-1)^{\nu \mathbf{1}} \\
&\quad \cdot \sum_{\mathbf{z} \in \mathbf{R}_m^s} \frac{V(1, \eta, \gamma, \mathbf{z})}{\mathbf{xb} + \nu \mathbf{b} + \mu_{\mathbf{z}}(1, \eta)} \sum_{j=1}^m (y_j + 1) b_j \mathbf{L}_{\mathbf{y} + \mathbf{1}_j, \mathbf{z}} \\
&\quad \cdot \left\{ \frac{1 - A(\eta + \mathbf{xb} + \nu \mathbf{b})}{\eta + \mathbf{xb} + \nu \mathbf{b}} - \frac{1 - A(\eta - \mu_{\mathbf{z}}(1, \eta))}{\eta - \mu_{\mathbf{z}}(1, \eta)} \right\} \Bigg],
\end{aligned}
\tag{4.86}
$$

where the function $V(1, \eta, \mathbf{z})$ is defined in (4.32). Moreover, for $\gamma \geq s$, $Re(\eta) > 0$, $l = 0, 1, 2, \cdots, s-2$;

$$
\begin{aligned}
&\int_0^\infty exp(-\eta t) P(C_t^* = l | C_0 = \gamma) dt \\
&= \sum_{\mathbf{x} \in \mathbf{R}_m^l} \Bigg[\sum_{k=l}^{s-1} \sum_{\mathbf{y} \in \mathbf{R}_m^k} \sum_{\nu \leq \mathbf{y} - \mathbf{x}} \binom{\mathbf{y}}{\mathbf{x}; \nu} (-1)^{\nu \mathbf{1}} \frac{1 - A(\eta + \mathbf{xb} + \nu \mathbf{b})}{\eta + \mathbf{xb} + \nu \mathbf{b}} \\
&\quad \cdot \sum_{i=1}^m p_i Z(1, \eta, \gamma; \mathbf{y} - \mathbf{1}_i) - \sum_{\mathbf{y} \in \mathbf{R}_m^{s-1}} \sum_{\nu \leq \mathbf{y} - \mathbf{x}} \binom{\mathbf{y}}{\mathbf{x}; \nu} (-1)^{\nu \mathbf{1}} \sum_{j=1}^m (y_j + 1) b_j \mathbf{L}_{\mathbf{y} + \mathbf{1}_j, \mathbf{z}} \\
&\quad \cdot \Bigg(\sum_{\mathbf{z} \in \mathbf{R}_m^s} \left(\frac{V_1(1, \eta, \gamma, \mathbf{z})}{-\mathbf{xb} - \nu \mathbf{b} - \mu_{\mathbf{z}}(1, \eta)} - \sum_{i=1}^a \frac{a_i(\mathbf{x}) V_2(1, \eta; \mathbf{x})}{(-\mathbf{xb} - \nu \mathbf{b} + w)^{a-i+1}} \right) \\
&\quad \cdot \frac{(1 - A(\eta + \mathbf{xb} + \nu \mathbf{b}))}{(\eta + \mathbf{xb} + \nu \mathbf{b})} + \sum_{\mathbf{z} \in \mathbf{R}_m^s} \frac{(\mathbf{xb} + \nu \mathbf{b})}{\mu_{\mathbf{z}}(1, \eta)(\mathbf{xb} + \nu \mathbf{b} + \mu_{\mathbf{z}}(1, \eta))} \\
&\quad \cdot \frac{(1 - A(\eta - \mu_{\mathbf{z}}(1, \eta)))}{\eta - \mu_{\mathbf{z}}(1, \eta)} V_1(1, \eta, \gamma; \mathbf{z}) + \sum_{\mathbf{z} \in \mathbf{R}_m^s} \sum_{i=1}^a (\mathbf{xb} + \nu \mathbf{b}) a_i(\mathbf{z}) V_2(1, \eta; \mathbf{z}) \\
&\quad \cdot \frac{1}{(i-1)!} \lim_{\xi \to -w} \frac{\partial^{i-1}}{\partial \xi^{i-1}} \frac{(1 - A(\eta - \xi))}{\xi(\mathbf{xb} + \nu \mathbf{b} + \xi)(\eta - \xi)} \Bigg) \Bigg].
\end{aligned}
\tag{4.87}
$$

An inversion of the transforms (4.86) and (4.87) will yield $P(C_t^* = l | C_0 = \gamma)$ for $l = 0, 1, \cdots, s-2$. The probabilities for $l = s-1$ and $l = s$ can be obtained from (4.83) and (4.84) with help of (4.47), (4.48), and (4.77).

For $\rho < 1$, we have for $l = 0, 1, 2, \cdots, s-2$;

$$\begin{aligned}
&P(C^* = l)\\
&= \sum_{\mathbf{x}\in\mathbf{R}_m^l} \left[\sum_{k=l}^{s-1} \sum_{\mathbf{y}\in\mathbf{R}_m^k} \sum_{\nu\leq\mathbf{y}-\mathbf{x}} \binom{\mathbf{y}}{\mathbf{x};\nu} (-1)^{\nu\mathbf{1}} \frac{1-A(\mathbf{x}\mathbf{b}+\nu\mathbf{b})}{\alpha(\mathbf{x}\mathbf{b}+\nu\mathbf{b})} \sum_{i=1}^{m} p_i \bar{Z}(\mathbf{y}-\mathbf{1}_i) \right.\\
&\quad - \sum_{\mathbf{y}\in\mathbf{R}_m^{s-1}} \sum_{\nu\leq\mathbf{y}-\mathbf{x}} \binom{\mathbf{y}}{\mathbf{x};\nu} (-1)^{\nu\mathbf{1}} \sum_{\mathbf{z}\in\mathbf{R}_m^s} \frac{\hat{V}(\mathbf{z})}{-\mathbf{x}\mathbf{b}-\nu\mathbf{b}-\tilde{\mu}_\mathbf{z}} \qquad (4.88)\\
&\quad \left. \cdot \sum_{j=1}^{m} (y_j+1) b_j \mathbf{L}_{\mathbf{y}+\mathbf{1}_j,\mathbf{z}} \left[\frac{(1-A(\mathbf{x}\mathbf{b}+\nu\mathbf{b}))}{\alpha(\mathbf{x}\mathbf{b}+\nu\mathbf{b})} \frac{(1-A(-\tilde{\mu}_\mathbf{z}))}{\alpha\tilde{\mu}_\mathbf{z}} \right] \right],
\end{aligned}$$

in accordance with equation (3.14) in [23]. The probabilities $P(C^* = l)$ for $l = s-1$ and $l = 2$ can be obtained from the steady-state version of (4.83) and (4.84), i.e.

$$P(C^* = s-1|C_0 = \gamma) = P(W^* = 0|C_0 = \gamma) - \sum_{j=0}^{s-2} P(C^* = l|C_0 = \gamma), \qquad (4.89)$$

$$P(C^* = s|C_0 = \gamma) = P(Q^* = 0|C_0 = \gamma) - P(W^* = 0|C_0 = \gamma), \qquad (4.90)$$

where $P(W^* = 0|C_0 = \gamma)$ and $P(Q^* = 0|C_0 = \gamma)$ can be obtained from the inversion of (4.49) and (4.78).

4.11 Numerical Examples

In this section, we give some examples of the distributions of interest studied in the previous sections. We restrict ourselves to the model $GI/H_2/s$ in which, as in [22], the elements of $\bigcup_{k=0}^{s-2} \mathbf{R}_2^k = \{(i,j)|i \geq 0, j \geq 0, i+j \leq s-2\}$ are numbered by using the one to one correspondence

$$(i,j) \to i+1+(s+i+j+1)(s-i-j-2)/2,$$

and the elements in $\mathbf{R}_2^{s-1} = \{(j-1, s-j)|j = 1, \cdots, s\}$ are numbered by the one to one correspondence $(j-1, s-j) \to j$.

In particular we consider the system $M/H_2/s$ for $s = 2$, in which the service time distribution function G is given by

$$G(x) = \begin{cases} 0.7(1-exp(-2.5x)) + 0.3(1-exp(-1.5x)) & , x \geq 0, \\ 0 & , x < 0. \end{cases} \qquad (4.91)$$

The mean of the service time is then given by 24/50. The Laplace-Stieltjes transform of the inter-arrival time is given by

$$A(\phi) = \frac{1.5}{\phi + 1.5}$$

thus the inter-arrival time has an exponential distribution with mean 2/3. The traffic intensity is 0.72.

4.11.1 Numerical results on the phase vectors

For the system under consideration, the system of equation (4.24) turns out to be a single equation since the set $\bigcup_{k=0}^{s-2} \mathbf{R}_m^k$ consists of one element, i.e. the vector $(0,0)$. We then can solve the system for $\eta = 0$ analytically to get $Z(r,0,\gamma)$, the generating function of $P(\mathbf{X}_n = (0,0)|C_0 = \gamma)$. We invert this generating function numerically by applying the numerical inversion algorithm described in Abate & Whitt [3], written in FORTRAN 90, to get $P(\mathbf{X}_n = (0,0)|C_0 = \gamma)$ for some velues of n.

The value of $P(\mathbf{X}_n = (0,0)|C_0 = \gamma)$ which is obtained from the inversion can be substituted into (4.21) to obtain the generating functions of $P(\mathbf{X}_n = (0,1)|C_0 = \gamma)$ and $P(\mathbf{X}_n = (1,0)|C_0 = \gamma)$. Again, we apply numerical inversion to get these probabilities.

The steady-state probabilities $P(\mathbf{X} = (0,0))$, $P(\mathbf{X} = (0,1))$, and $P(\mathbf{X} = (1,0))$ can be obtained from the equations (4.38) and (4.37).

In figure 4.2 we give the probabilities of the time-dependent phase vectors $P(\mathbf{X}_n = (0,0)|C_0 = \gamma)$, $P(\mathbf{X}_n = (0,1)|C_0 = \gamma)$, and $P(\mathbf{X}_n = (1,0)|C_0 = \gamma)$ for some values of n and its steady-state probabilities, for $\gamma = 0$.

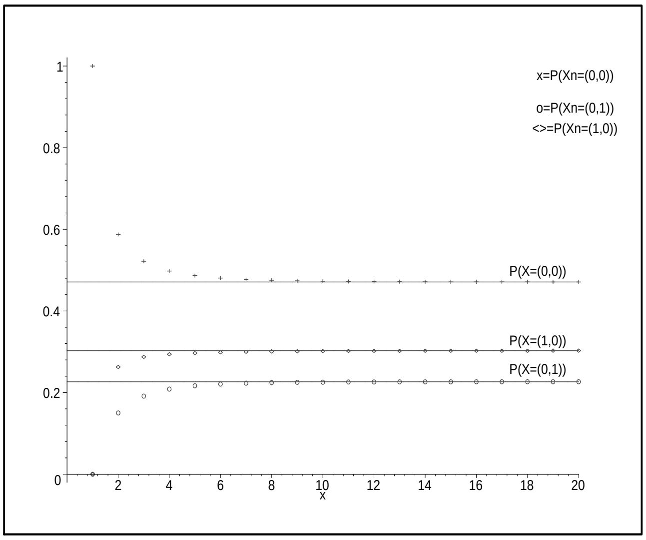

Figure 4.2: $P(\mathbf{X}_n = \mathbf{x})$ for some n with $\mathbf{X}_1 = (0,0)$

4.11.2 Numerical results on some distributions of interest

We give some numerical results on the distributions we discussed in sections 4.5 to 4.8 for some values of γ. The transforms of all distributions in those sections involve the generating function $\bar{\mathcal{Z}}(r,\eta,0,\gamma)$, where from (4.21) its explicit expression can be obtained by first solving the system (4.24) and factoring the matrix $\mathbf{H}(r, \eta, \phi)$, and then substituting the results into (4.20) and Lemma 4.3.1.

The Laplace-Stieltjes transform of the actual waiting time in steady state is given by the equation (4.44). We can invert this transform analytically to get the distribution. For the time-dependent case, the Laplace-Stieltjes transform of the actual waiting time for $\gamma < s$ or $(\gamma \geq s, \mathbf{X}_1 \notin \mathbf{R}_m^{s-1})$ is given by the generating function (4.43), and for $\gamma \geq s, \mathbf{X}_1 \in \mathbf{R}_m^{s-1}$ it is given by the generating function (4.39). Since these double transforms are rational functions with respect to one variable, first we invert it analytically and then invert it numerically by applying the numerical inversion algorithm for generating function in [3]. Some results on the distribution of the actual waiting time for some values of γ can be seen in figures 4.3 to 4.6. The time-dependent distributions in figures 4.3 to 4.5 are obtained by inverting the generating function (4.43) analytically and then numerically. The time-dependent distribution in figure 4.6 is obtained from the generating function (4.39) in the same way.

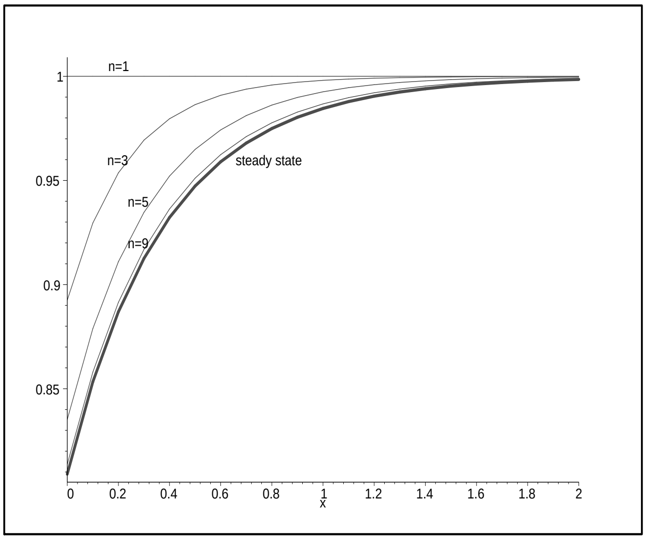

Figure 4.3: $P(W_n \leq x | C_0 = 0)$ for some n and $P(W \leq x)$, $\mathbf{X}_1 = (0, 0)$.

The Laplace-Stieltjes transform of the virtual waiting time in steady state is given by the equation (4.49). This transform can be inverted analytically to get the distribution. For the time-dependent distribution, we invert the double Laplace-Stieltjes transform (4.47) in the same way as for the actual waiting time. Some results on the distribution of the virtual waiting time can be seen in figures 4.7 to 4.10.

The steady-state expectations of the queue length and of the number of customers at arrival epochs are given by (4.57) and (4.69). The transform of the time-dependent

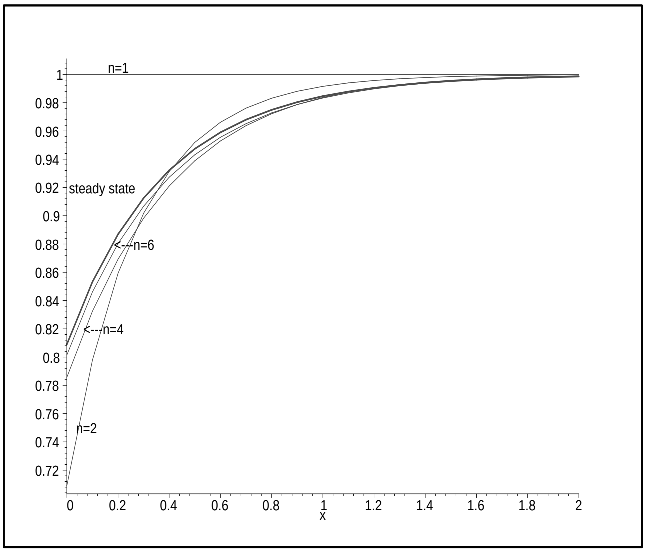

Figure 4.4: $P(W_n \leq x|C_0 = 1)$ for some n and $P(W \leq x)$, $\mathbf{X}_1 = (0, 1)$

expectation of the queue length at arrival epochs for $\gamma < s$ and for $\gamma \geq s$ are given by the generating functions (4.56) and (4.60), respectively. The time-dependent transforms of the number of customers for $\gamma < s$ and for $\gamma \geq s$ are given by (4.56) and (4.68), respectively. We perform a numerical inversion of these transforms, and the results can be seen in figures 4.11 to 4.12.

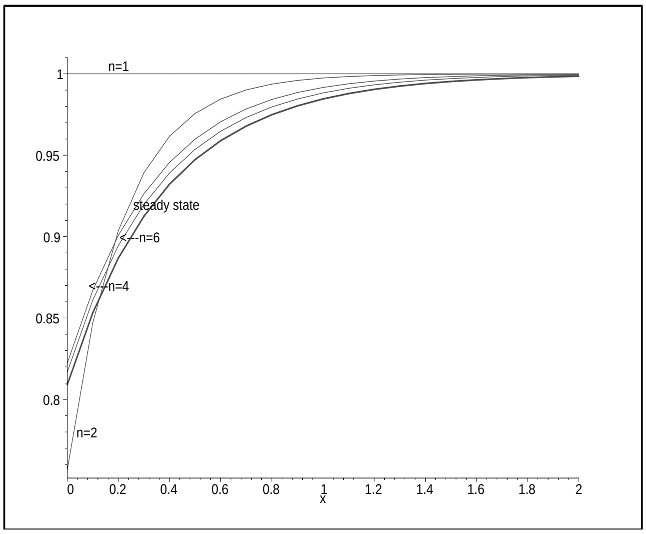

Figure 4.5: $P(W_n \leq x|C_0 = 1)$ for some n and $P(W \leq x)$, $\mathbf{X}_1 = (1, 0)$

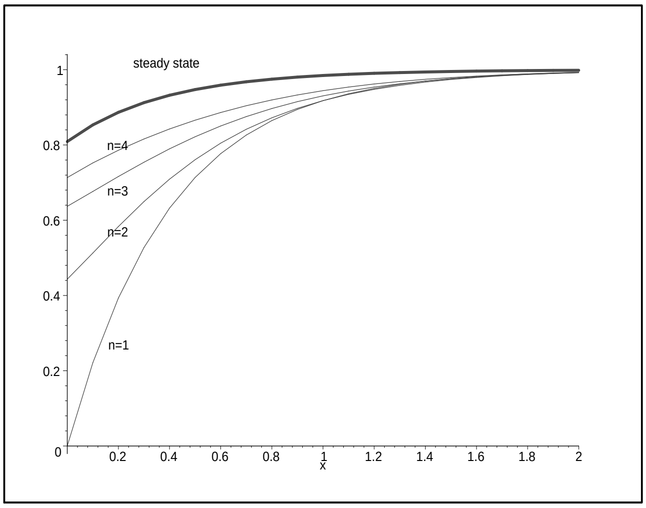

Figure 4.6: $P(W_n \leq x|C_0 = 2)$ for some n and $P(W \leq x)$, $\mathbf{X}_1 = (0, 1)$

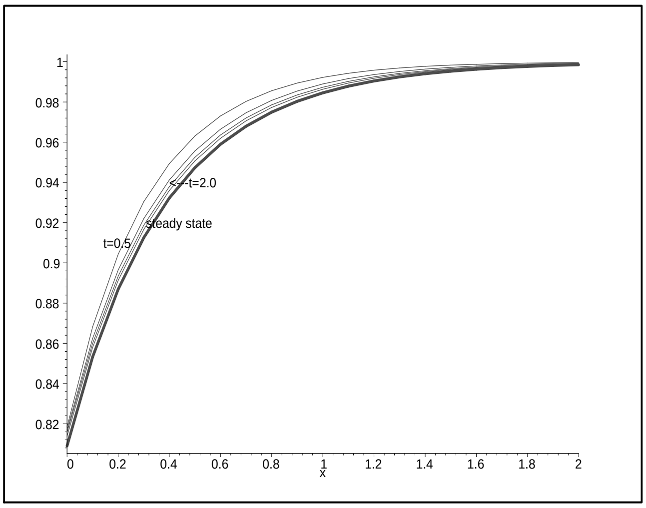

Figure 4.7: $P(W_t^* \leq x | C_0 = 0)$ for some t and $P(W^* \leq x)$, $\mathbf{X}_1 = (0, 0)$.

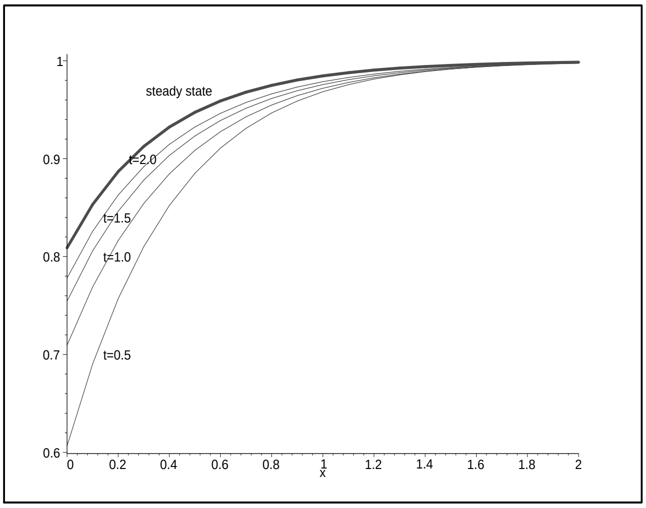

Figure 4.8: $P(W_t^* \leq x | C_0 = 1)$ for some t and $P(W^* \leq x)$, $\mathbf{X}_1 = (0, 1)$.

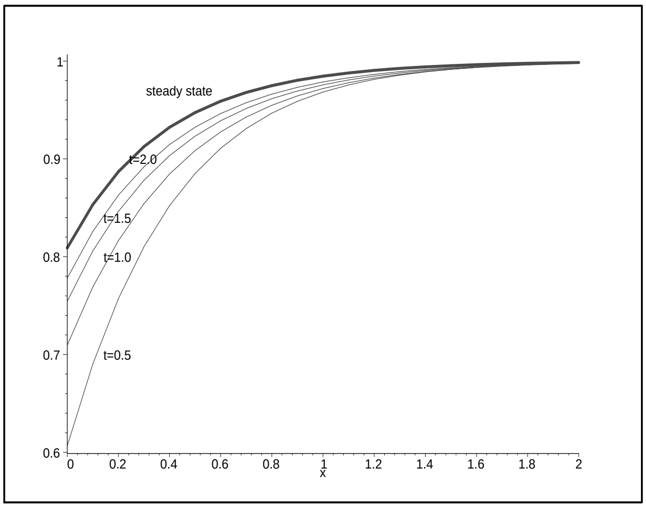

Figure 4.9: $P(W_t^* \leq x|C_0 = 1)$ for some t and $P(W^* \leq x)$, $\mathbf{X}_1 = (1, 0)$.

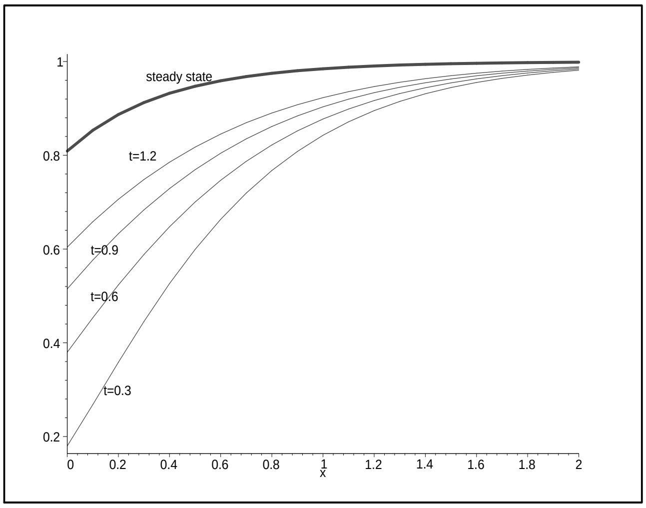

Figure 4.10: $P(W_t^* \leq x|C_0 = 2)$ for some t and $P(W^* \leq x)$, $\mathbf{X}_1 = (0, 1)$.

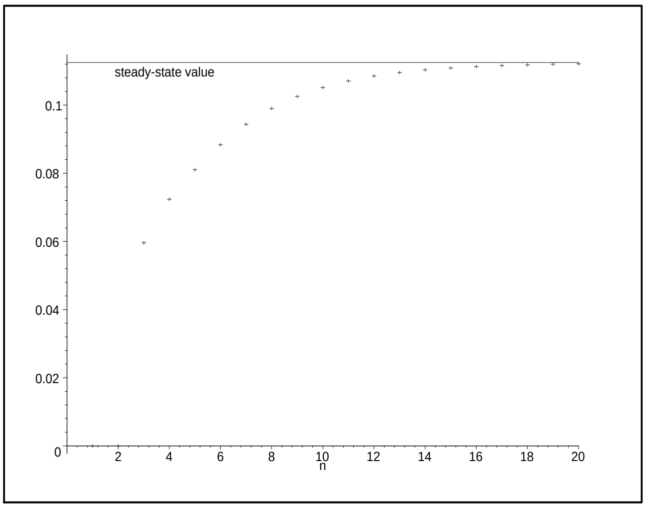

Figure 4.11: $E(Q_n|C_0 = 1)$ for some n and $E(Q)$, $\mathbf{X}_1 = (1, 0)$.

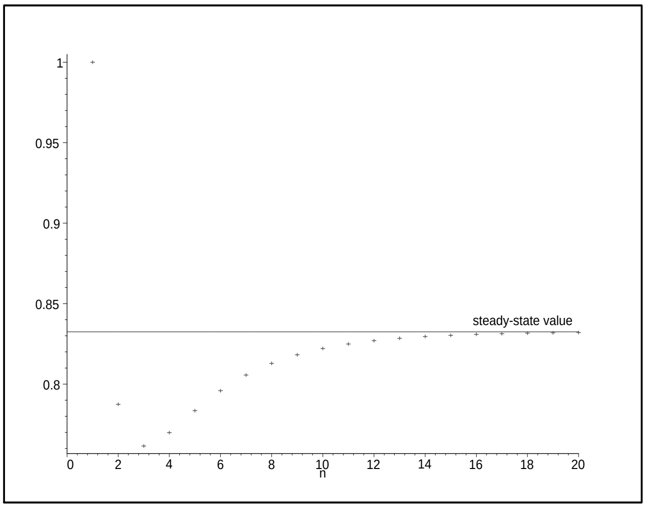

Figure 4.12: $E(C_n|C_0 = 1)$ for some n and $E(C)$, $\mathbf{X}_1 = (1, 0)$.

Chapter 5

Markovian Fluid Flow Model

5.1 Introduction

In this chapter we study a fluid flow model in which the rate of the input process $\{a_t\}$ depends on the state of a finite-state continuous-time irreducible Markov chain $\{J_t\}$ with state space $\mathcal{N} = \{1, 2, \ldots, N\}$. More precisely, the slope of the process $\{a_t\}$ is constant between transitions of $\{J_t\}$ and equal to c_i when $\{J_t\}$ is in state i. The input flows into an infinite buffer that has maximal output rate c, and initially has a content v. We define the net input S_t up to time t as the difference between the total traffic received up to time t and the maximal output traffic up to time t. Then the rate of the net input process $\{S_t\}$ is also constant between transitions of $\{J_t\}$, and is equal to $r_i = c_i - c$. The buffer content V_t at time $t \geq 0$ is found from S_t by restricting it to nonnegative values, i.e. by applying the reflection operation

$$V_t = v + S_t + I_t$$

where

$$I_t = \max\left\{-v - \inf_{0\leq\nu\leq t} S_\nu, 0\right\}.$$

Let $T_0 = 0$ and $T_1, T_2, \ldots$ be the transition epochs of the process $\{J_t\}$ with $T_1 > 0$. We define $W_0 = v$ and for $n = 1, 2, \ldots$, $W_n = V_{T_n}$ and $X_n = J_{T_n^-}$, where T_n^- is the state of $\{J_t\}$ just before transition epochs. We are interested in the probability distribution of the buffer content in steady state and as well as for the time-dependent case, for which the distribution functions at time $t \geq 0$ are denoted by

$$F_{ij}(x, t, v) = P(V_t \leq x, J_t = j | X_1 = i, V_0 = v), \quad i, j \in \mathcal{N}.$$

This model and its special cases, as well as generalizations of it, have been studied by several authors. Many of them studied the special case in which the system consists of a finite number of sources (input lines), each of which alternates between the 'on' and the 'off' state. All on-periods have the same exponential distribution and, similarly, all off-periods. Moreover all on- and off-periods are independent random variables. The sources are merged into a single data stream via a switch with a buffer.

When a source is 'on' it feeds data into the switch with rate b, while the buffer has a maximal output rate a. The net input S_t up to time t is the difference between the total traffic received up to time t and at, i.e. the maximal output traffic up to time t. We see that this system is a special case of the fluid flow model, where $\{J_t\}$ is the number of active sources at time t and $r_i = ib - a$ is the net input rate when there are i active sources. Moreover we see that $\{J_t\}$ is a Birth and Death process.

This special case has been studied by Anick, Mitra, and Sondhi(AMS) [4], and Kosten [31]. They determined the distribution function of the buffer content in steady state by deriving the Kolmogorov forward equations for the Markov chain $\{(V_t, J_t), t \geq 0\}$ and analyzing this system of equations. Furthermore, in [4] and [31], $G(u)$, -the steady-state probability that the buffer content exceeds a certain level u- is studied. In AMS [4] an asymptotic formula for $G(u)$, is given, of the form $Ce^{-\alpha u}$, where C is a constant and α is a positive parameter. An approximation method to calculate $G(u)$ is also given in this paper. In [31] the same asymptotic formula for $G(u)$ is derived, and a simulation method is used to find the constant C.

The model in which $\{J_t\}$ is a general Markov chain has been studied by Regterschot [38], Asmussen [8], Rogers [40], and Pacheco & Prabhu [37]. The first three authors consider the model with an initially empty buffer, and the last ones studied the non-empty case. With a decomposition method, Asmussen [8] proved that the steady state distribution of the buffer content is of phase type and proposed an algorithm to compute its phase generator U. Regterschot [38], Rogers [40] and Pacheco & Prabhu [37] use Wiener-Hopf factorization of a certain matrix. As a main result, they obtain the steady state distribution function of the buffer content. In [38], an explicit formula for the steady state distribution function is obtained not only for continuous time but also at transition epochs. Furthertmore, an explicit formula for $G(u)$ is given.

The transient behavior of the buffer content for the present model is studied by Tanaka et al. [42]. They refer to the model as *Markov modulated input rate(MMIR) model,* since the input is generated by a Markov modulated process. The Laplace transform of the joint distribution of the buffer content and the state of the input process is derived. By analyzing the properties of the eigenvalues and eigenvectors of a certain matrix they found an explicit expression for this transform.

In solving the present problem, we use the same technique as in [38] but we give some corrections for the factorization. In section 5.2 we consider the process $\{(W_n,\ T_n,\ X_n)\}$ and derive Wiener-Hopf type equations for the transform of the joint distribution of $\{(W_n, T_n, X_n)\}$. Then in section 5.3, we solve this system of equation with Wiener-Hopf factorization, which boils down to finding some eigenvalues and eigenvectors of a certain matrix and solving some matrix equations. The factorization is similar to the one in [38] but since we have initially a non-empty buffer we need to decompose a certain matrix as an additional step to complete the solution. In section 5.4 we derive an explicit expression for the steady state distribution function of the buffer content at transition epochs by considering the process $\{(W_n, X_n)\}$. This result is found from the generating function of transforms in section 5.3 using Abel's limit theorem, and with the factorization we obtain an explicit expression for the steady state distribution function after we analytically invert its transform. In section 5.5 we derive the double Laplace transform of the time-dependent distribution function of the buffer content and again after a limiting operation we get the Laplace transform of the steady state distribution. The explicit expression for the steady state distribution can be found by inverting this transform, as shown in subsection 5.5.1. Since the double Laplace transform consists of terms involving multiplication of exponential and rational terms in one variable only, to get the time-dependent distribution of the buffer content we first invert it analytically. The result of this analytical inversion can be found in subsection 5.5.2, and it shows a structure quite similar to the one in [42].

We study the behavior of the time-dependent distribution of V_t as t increases, by referring to the relaxation time (see Blanc and van Doorn[14]), a measure of the speed of convergence to the steady-state distribution. In Tanaka, *et al.*[42], the analysis ends with a conjecture stating that the relaxation time for the present model depends on the generator of the underlying Markov chain, but not on the rates of the input flow. We study the relaxation time in subsection 5.5.3, and show that the relaxation time also depends on the net input rates $r_i, i \in \mathcal{N}$.

We implement the numerical inversion algorithm proposed in Abate and Whitt[3] to get the desired distributions. Some examples for these distributions can be found in section 5.6.

We will use the following notations: $x^+ = \max(0, x)$, and $x^- = \min(0, x)$. *1* is the indicator function, $\mathbf{1}$ is the N-dimensional column vector with all components equal to 1, $\mathbf{1}_i$ is the $\bar{K}$-dimensional column vector with i-th component 1 and all other component equal to 0, where $\bar{K}$ is an integer defined in section 5.2. $\mathbf{I}$ is the identity matrix, $\mathbf{I}_{kl}$ is the $k \times l$-matrix with elements δ_{ij}, where δ_{ij} is Kronecker's delta, i.e., $\delta_{ij} = 0$, for $i \neq j$, and $\delta_{jj} = 1$. If $\mathbf{A}$ is an $N \times N$-dimensional matrix, the i-th column of $\mathbf{A}$ is denoted by $\mathbf{A}^i$, and the i-th row of $\mathbf{A}$ is denoted by $\mathbf{A}_i$.

5.2 System of Wiener-Hopf-type equations

Let $\mathbf{Q}$ be the infinitesimal generator of the Markov chain $\{J_t\}$ with elements Q_{ij}, and let $\mathbf{P}$ be the transition probability matrix of $\{X_n\}$ with elements P_{ij}. We assume that the matrix $\mathbf{Q}$ is indecomposable. Define $q_i = -Q_{ii} = \sum_{i \neq j} Q_{ij}$, $i, j \in \mathcal{N}$. Let $\mathbf{q} = \text{diag}(q_1, \ldots, q_N)$ be the $N \times N$ dimensional diagonal matrix with elements q_i. It follows that

$$\mathbf{P} = \mathbf{q}^{-1}(\mathbf{Q} + \mathbf{q}).$$

The stationary probabilities $\lim_{t\to\infty} P(J_t = i)$ are denoted by π_i, $i \in \mathcal{N}$ and π denotes the $N-$ dimensional row vector with components π_i. We assume that $\sum_{i=1}^N \pi_i r_i < 0$ to ensure stability. The stationary probabilities $\lim_{n\to\infty} P(X_n = i)$ are denoted by γ_i, $i \in \mathcal{N}$ and $\boldsymbol{\gamma}$ denotes the N-dimensional row vector with components γ_i. It follows that

$$\gamma_i = \frac{\pi_i q_i}{\boldsymbol{\pi q 1}}, \quad i = 1, 2, \cdots, N. \tag{5.1}$$

The traffic intensity ρ, i.e. the ratio of the average input rate and the maximal output rate , is $\rho = \sum_{i=1}^N \pi_i c_i / c$.

We assume that for $i \in \mathcal{N}$, $c_i \neq c$ so that $r_i \neq 0$ for $i \in \mathcal{N}$. Let $R^- = \{i | r_i < 0, i = 1, \ldots, N\}$ and $R^+ = \{i | r_i > 0, i = 1, \ldots, N\}$. Suppose that $|R^-| = \bar{K}$. This implies that $|R^+| = N - \bar{K}$. Let $\mathbf{r} = \text{diag}(r_1, \ldots, r_N)$. Without loss of generality, suppose that $R^- = \{1, 2, \ldots, \bar{K}\}$.

Let $T_0 = 0$ and $T_1, T_2, \ldots$ be the transition epochs of the process $\{J_t\}$ with $T_1 > 0$, and let $A_n = T_n - T_{n-1}$ be the inter-jump time, $n = 1, 2, \ldots$. Define $R_n = \sum_k r_k \mathit{1}\{X_n = k\}$, we then have the relation

$$W_{n+1} = [W_n + R_{n+1}A_{n+1}]^+.$$

Define for $Re(\eta) \geq 0, Re(\phi) \geq 0$,

$$Z_i^0(\phi, \eta, v) = E\left(e^{-\phi W_1 - \eta T_1} \mathit{1}(X_1 = i) | X_1 = i, V_0 = v\right), \tag{5.2}$$

and define for ($|z| < 1, Re(\eta) \geq 0, Re(\phi) \geq 0$) or ($|z| \leq 1, Re(\eta) > 0, Re(\phi) \geq 0$) or ($|z| \leq 1, Re(\eta) \geq 0, Re(\phi) > 0$)

$$Z_{ij}(z, \phi, \eta, v) = \sum_{n=1}^{\infty} z^n E\left(e^{-\phi W_n - \eta T_n} \mathit{1}(X_n = j) | X_1 = i, V_0 = v\right).$$

Define for $Re(\phi) \leq 0$,

$$\begin{aligned} V_{ij}(z, \phi, \eta, v) &= \sum_{n=1}^{\infty} z^{n+1} E\left(\left(1 - e^{-\phi [W_n + R_{n+1}A_{n+1}]^-}\right)\right. \\ &\qquad \left. e^{-\eta(T_n + A_{n+1})} \mathit{1}(X_n = j) | X_1 = i, V_0 = v\right), \end{aligned}$$

and for $Re(\phi) = 0$,

$$G_{ij}(\phi, \eta) = E\left(e^{-(r_j\phi + \eta)A_{n+1}} \mathit{1}(X_{n+1} = j) | X_n = i\right).$$

Let $\mathbf{Z}(z, \phi, \eta, v), \mathbf{V}(z, \phi, \eta, v)$ and $\mathbf{G}(\phi, \eta)$ be $N \times N-$ matrices with elements $Z_{ij}(z, \phi, \eta, v)$, $V_{ij}(z, \phi, \eta, v)$ and $G_{ij}(\phi, \eta)$ respectively. We then obtain the following system of Wiener-Hopf equations.

Theorem 5.2.1
For $Re(\phi) = 0$ and ($|z| \leq 1, Re(\eta) > 0$) or ($|z| < 1, Re(\eta) \geq 0$) we have

$$\mathbf{Z}(z, \phi, \eta, v)(\mathbf{I} - z\mathbf{G}(\phi, \eta)) = z\mathbf{Z}^0(\phi, \eta, v) + \mathbf{V}(z, \phi, \eta, v), \tag{5.3}$$

where

$$\mathbf{Z}^0(\phi, \eta, v) = diag(Z_1^0(\phi, \eta, v), Z_2^0(\phi, \eta, v), \ldots, Z_N^0(\phi, \eta, v))$$

with

$$Z_i^0(\phi, \eta, v) = \begin{cases} e^{-\phi v} \frac{q_i}{\phi r_i + \eta + q_i} & , \text{ if } r_i > 0 \\ \\ \frac{q_i}{\phi r_i + \eta + q_i}\left[e^{-\phi v} - e^{(\eta + q_i)v/r_i}\right] + \frac{q_i e^{(\eta+q_i)v/r_i}}{\eta + q_i} & , \text{ if } r_i < 0. \end{cases} \tag{5.4}$$

Proof. By the identity

$$e^{-\phi x^+} = e^{-\phi x} + 1 - e^{-\phi x^-}$$

we have for $Re(\phi) = 0, Re(\eta) \geq 0$

$$\begin{aligned}
& E\left(e^{-\phi W_{n+1}-\eta T_{n+1}} \mathit{1}(X_{n+1}=j)|X_1=i, V_0=v\right) \\
= \;& E\left(e^{-\phi[W_n+R_{n+1}A_{n+1}]^+} e^{-\eta(T_n+A_{n+1})} \mathit{1}(X_{n+1}=j)|X_1=i, V_0=v\right) \\
= \;& E\left(e^{-\phi(W_n+R_{n+1}A_{n+1})} e^{-\eta(T_n+A_{n+1})} \mathit{1}(X_{n+1}=j)|X_1=i, V_0=v\right) \\
& +E\left(\left(1-e^{-\phi[W_n+R_{n+1}A_{n+1}]^-}\right) e^{-\eta(T_n+A_{n+1})} \mathit{1}(X_{n+1}=j)|X_1=i, V_0=v\right) \\
= \;& E\left(e^{-\phi W_n-\eta T_n-(\phi R_{n+1}+\eta)A_{n+1}} \mathit{1}(X_{n+1}=j)|X_1=i, V_0=v\right) \\
& +E\left(\left(1-e^{-\phi[W_n+R_{n+1}A_{n+1}]^-}\right) e^{-\eta(T_n+A_{n+1})} \mathit{1}(X_{n+1}=j)|X_1=i, V_0=v\right).
\end{aligned} \tag{5.5}$$

Moreover

$$\begin{aligned}
& E\left(e^{-\phi W_n-\eta T_n-(\phi R_{n+1}+\eta)A_{n+1}} \mathit{1}(X_{n+1}=j)|X_1=i, V_0=v\right) \\
= \;& \sum_{k=1}^{N} E\left(e^{-\phi W_n-\eta T_n} \mathit{1}(X_n=k)|X_1=i, V_0=v\right) \\
& .E\left(e^{-(\phi R_{n+1}+\eta)A_{n+1}} \mathit{1}(X_{n+1}=j)|X_n=k\right) \\
= \;& \sum_{k=1}^{N} E\left(e^{-\phi W_n-\eta T_n} \mathit{1}(X_n=k)|X_1=i, V_0=v\right) \\
& .E\left(e^{-(\phi r_j+\eta)A_{n+1})} \mathit{1}(X_{n+1}=j)|X_n=k\right) \\
= \;& \sum_{k=1}^{N} E\left(e^{-\phi W_n-\eta T_n} \mathit{1}(X_n=k)|X_1=i, V_0=v\right) G_{kj}(\phi,\eta).
\end{aligned}$$

Substituting this into (5.5) we get

$$\begin{aligned}
& E\left(e^{-\phi W_{n+1}-\eta T_{n+1}} \mathit{1}(X_{n+1}=j)|X_1=i, V_0=v\right) \\
= \;& \sum_{k=1}^{N} E\left(e^{-\phi W_n-\eta T_n} \mathit{1}(X_n=k)|X_1=i, V_0=v\right) G_{kj}(\phi,\eta) \\
& +E\left(\left(1-e^{-\phi[W_n+R_{n+1}A_{n+1}]^-}\right) e^{-\eta(T_n+A_{n+1}} \mathit{1}(X_{n+1}=j)|X_1=i, V_0=v\right)
\end{aligned}$$

If we multiply by z^{n+1} and sum over n this yields for $(Re(\phi) = 0, Re(\eta) \geq 0, |z| < 1)$ or $(Re(\phi) = 0, Re(\eta) > 0, |z| \leq 1)$,

$$Z_{ij}(z,\phi,\eta,v) - z\delta_{ij}Z_i^0(\phi,\eta,v) = z\sum_{k=1}^{N} Z_{ik}(z,\phi,\eta,v)G_{kj}(\phi,\eta) + V_{ij}(z,\phi,\eta,v)$$

and we get equation (5.3). It remains to verify the expression for $Z_i^0(\phi,\eta,v)$. To this end we note that

$$\begin{aligned}
Z_i^0(\phi,\eta,v) &= E\left(e^{-\phi W_1-\eta T_1}\mathit{1}(X_1=i)|X_1=i,V_0=v\right)\\
&= \int_0^\infty E\left(e^{-\phi[v+r_iT_1]^+-\eta T_1}|T_1=u\right)q_ie^{-q_iu}du\\
&= \int_0^\infty e^{-\phi[v+r_iu]^+-\eta u}q_ie^{-q_iu}du.
\end{aligned}$$

So, for $r_i < 0$,

$$\begin{aligned}
Z_i^0(\phi,\eta,v) &= \int_0^{-v/r_i} e^{-\phi v-(\phi r_i+\eta)u}q_ie^{-q_iu}du + \int_{-v/r_i}^\infty e^{-\eta u}q_ie^{-q_iu}du\\
&= e^{-\phi v}q_i\left[\frac{-e^{(\phi r_i+\eta+q_i)v/r_i}+1}{\phi r_i+\eta+q_i}\right] + \frac{q_ie^{(\eta+q_i)v/r_i}}{\eta+q_i},
\end{aligned}$$

and for $r_i > 0$,

$$\begin{aligned}
Z_i^0(\phi,\eta,v) &= \int_0^\infty e^{-\phi v-(\phi r_i+\eta)u}q_ie^{-q_iu}du\\
&= e^{-\phi v}\frac{q_i}{\phi r_i+\eta+q_i}.
\end{aligned}$$

■

The system (5.3) is the generalization of the system in [38], and will be solved by applying the Wiener-Hopf factorization technique.

5.3 Solution of the system of Wiener-Hopf equations

In order to solve the system (5.3) we first factorize the symbol

$$\mathbf{H}(z,\phi,\eta) = \mathbf{I} - z\mathbf{G}(\phi,\eta), \tag{5.6}$$

i.e. for $Re(\phi) = 0$, we try to find a factorization

$$\mathbf{H}(z,\phi,\eta) = \mathbf{H}^+(z,\phi,\eta)\mathbf{H}^-(z,\phi,\eta)$$

where

$\mathbf{H}^+(z,\phi,\eta)$ is analytic for $Re(\phi) > 0$, and continuous and bounded for $Re(\phi) \geq 0$, and non-singular in $Re(\phi) > 0$.

$\mathbf{H}^-(z,\phi,\eta)$ is analytic for $Re(\phi) < 0$, and continuous and bounded for $Re(\phi) \leq 0$, and non-singular in $Re(\phi) < 0$.

In Arjas [6], the probabilistic interpretation of these factors is given.

To find $\mathbf{H}^{+}(z,\phi,\eta)$ and $\mathbf{H}^{-}(z,\phi,\eta)$ first we consider the following .

With the definition of q and r we can write the matrix $\mathbf{G}(\phi,\eta)$ as

$$\mathbf{G}(\phi,\eta) = \mathbf{q}^{-1}(\mathbf{Q}+\mathbf{q})\mathbf{q}(\mathbf{q}+\phi\mathbf{r}+\eta\mathbf{I})^{-1} = \mathbf{P}\mathbf{q}(\mathbf{q}+\phi\mathbf{r}+\eta\mathbf{I})^{-1}. \tag{5.7}$$

Let $\alpha_i(\eta) = (\eta+q_i)/r_i$, $i = 1,2,\ldots,N$. Define $N \times N-$dimensional matrices

$$\begin{aligned}
\boldsymbol{\alpha}(\eta) &= \mathrm{diag}(\alpha_1(\eta),\ldots,\alpha_N(\eta)),\\
\boldsymbol{\alpha} = \boldsymbol{\alpha}(0) &= \mathrm{diag}\left(\frac{q_1}{r_1},\ldots,\frac{q_N}{r_N}\right),\\
\mathbf{L}(z,\phi,\eta) &= \phi\mathbf{I}+\boldsymbol{\alpha}(\eta) - z\boldsymbol{\alpha} - z\mathbf{r}^{-1}\mathbf{Q},
\end{aligned}$$

and

$$\mathbf{M}(\phi,\eta) = \boldsymbol{\alpha}(\eta)+\phi\mathbf{I}.$$

It follows that

$$\mathbf{H}(z,\phi,\eta) = \boldsymbol{\alpha}^{-1}\mathbf{L}(z,\phi,\eta)\boldsymbol{\alpha}\mathbf{M}^{-1}(\phi,\eta). \tag{5.8}$$

From (5.8) we have that

$$\det\mathbf{L}(z,\phi,\eta) = \det\mathbf{H}(z,\phi,\eta)\det\mathbf{M}(\phi,\eta). \tag{5.9}$$

Proposition 5.3.1

1. *The poles of* $\det\mathbf{H}(z,\phi,\eta)$ *are the zeros of* $\det\mathbf{M}(\phi,\eta)$,
2. *The zeros of* $\det\mathbf{L}(z,\phi,\eta)$ *are the zeros of* $\det\mathbf{H}(z,\phi,\eta)$.

Proof. It is clear that the zeros of $\det\mathbf{M}(\phi,\eta)$ are $-\alpha_1(\eta)$, $-\alpha_2(\eta),\cdots,-\alpha_N(\eta)$. For $i=1,\cdots,\bar{K}$ these zeros lie in the right half-plane $Re(\phi)>0$ and for $i=\bar{K}+1,\cdots,N$ they lie in the left half-plane $Re(\phi)<0$.

Since $\det\mathbf{L}(z,\phi,\eta)$ does not have any pole, it follows from (5.9) that the poles of $\det\mathbf{H}(z,\phi,\eta)$ are the zeros of $\det\mathbf{M}(\phi,\eta)$. This proves part 1 of the proposition.

It also follows from (5.9) that the zeros of $\det\mathbf{L}(z,\phi,\eta)$ are the zeros of $\det\mathbf{H}(z,\phi,\eta)$ or the zeros of $\det\mathbf{M}(\phi,\eta)$. Since from (5.6) and (5.7) we see that $\det\mathbf{H}(z,\phi,\eta)$ has exactly N poles, then $\det\mathbf{L}(z,\phi,\eta)$ does not have any zero in common with $\det\mathbf{M}(\phi,\eta)$. This proves part 2 of the proposition.

■

Based on the proposition, we consider the following lemma, which has been proven in [38]. We rewrite the proof of part 2 since some intermediate results in this proof will be used in section 5.4 and thereafter.

Lemma 5.3.1
With respect to ϕ,

1. *for $(z,\eta) \neq (1,0)$, $\det \mathbf{L}(z,\phi,\eta)$ has $\bar{K}$ zeros in the right half-plane $Re(\phi) > 0$ and has $N-\bar{K}$ zeros in the left half-plane $Re(\phi) < 0$, and $\det \mathbf{L}(z,\phi,\eta) \neq 0$ on the imaginary axis $Re(\phi) = 0$.*

2. *$\det \mathbf{L}(1,\phi,0)$ has $\bar{K}-1$ zeros in the right half- plane $Re(\phi) > 0$ and a simple zero at $\phi = 0$, and has $N-\bar{K}$ zeros in the left half-plane $Re(\phi) < 0$.*

Proof. See [38] for the proof of part 1.

Let $\mu_1(z,\eta), \cdots, \mu_{\bar{K}}(z,\eta)$ be the zeros of $\det \mathbf{L}(z,\phi,\eta)$ in the right half-plane $Re(\phi) > 0$, and let $\mu_{\bar{K}+1}(z,\eta), \cdots, \mu_N(z,\eta)$ be the zeros of $\det \mathbf{L}(z,\phi,\eta)$ in the left half-plane $Re(\phi) < 0$. We consider the situation for $\eta = 0$. Now,

$$\mathbf{L}(z,\phi,0) = \phi\mathbf{I} + (1-z)(\mathbf{r}^{-1}\mathbf{Q} + \boldsymbol{\alpha}) - \mathbf{r}^{-1}\mathbf{Q}.$$

Define the matrix $\mathbf{L}^*(z,\phi)$ as a matrix obtained from $\mathbf{L}(z,\phi,0)$ by adding all columns to the first column so that

$$L^*_{i1}(z,\phi) = \phi + (1-z)\alpha_i, \quad i = 1,2,\ldots,N.$$

Since $L^*_{i1}(1,0) = 0, \quad i = 1,2,\ldots,N$, we have $\det \mathbf{L}^*(1,0) = \det \mathbf{L}(1,0,0) = 0$. With the implicit function theorem we can define the function $\mu(z)$ uniquely by $\mu(1) = 0$ and

$$\det \mathbf{L}^*(z,\mu(z)) = \det \mathbf{L}(z,\mu(z),0) = 0.$$

We consider this function for z close to 1, so

$$\mu(z) = -(1-z)\mu'(1) + o(1-z), \quad z \uparrow 1.$$

It follows that

$$\mathbf{L}(z,\mu(z),0) = (1-z)(\boldsymbol{\alpha} - \mu'(1)\mathbf{I}) - z\mathbf{r}^{-1}\mathbf{Q} + o(1-z), \quad z \uparrow 1$$

and

$$L^*_{i1}(z,\mu(z)) = (1-z)(\alpha_i - \mu'(1)) + o(1-z), \quad z \uparrow 1.$$

Since $\det \mathbf{L}^*(z,\mu(z)) = 0$, we have

$$0 = \lim_{z\uparrow 1} \frac{1}{1-z} \det \mathbf{L}^*(z,\mu(z)) = \det \mathbf{L}^o,$$

where for $i \in \mathcal{N}, j \in \mathcal{N}$,

$$L^o_{ij} = -r_i^{-1}Q_{ij},$$

and

$$L^\circ_{i1} = \alpha_i - \mu'(1). \tag{5.10}$$

Since $\det \mathbf{L}^o = 0$ and $\pi\mathbf{r}(\mathbf{r}^{-1}\mathbf{Q}) = 0$, we have

$$\sum_{i=1}^{N} \pi_i r_i L^o_{i1} = 0,$$

or with (5.10)

$$\mu'(1) = \sum_{i=1}^{N} \pi_i q_i / \sum_{i=1}^{N} \pi_i r_i = \pi\mathbf{q1}/\pi\mathbf{r1}$$

so that

$$\mu(z) = -(1-z)\frac{\pi\mathbf{q1}}{\pi\mathbf{r1}} + o(1-z), \quad z \uparrow 1. \tag{5.11}$$

For $z \uparrow 1$ one of the $\bar{K}$ zeros of $\det \mathbf{L}(z, \mu(z), 0)$ in the right half-plane $Re(\phi) > 0$ tends to 0 if and only if $\sum_{i=1}^{N} \pi_i r_i = \pi\mathbf{r1} < 0$. The latter condition corresponds to traffic intensity $\rho < 1$.

■

We next impose the following condition.

Condition 5.3.1
For $(z, \eta) = (1, 0)$ *and* $|z| \le 1, Re(\eta) > 0, -\alpha_1(\eta), \ldots, -\alpha_N(\eta)$ *and* $\mu_1(z,\eta), \ldots, \mu_N(z,\eta)$ *are all distinct, possibly with the exception of a set of isolated points.*

For $i = 1, \ldots, N$ let $\mathbf{E}^i(z, \eta)$ be a (non unique) nonzero column vector satisfying

$$\mathbf{L}(z, \mu_i(z,\eta), \eta)\mathbf{E}^i(z, \eta) = 0, \tag{5.12}$$

and let $\mathbf{E}(z, \eta)$ be the $N \times N$-matrix with ith column $\mathbf{E}^i(z, \eta)$. Let $\mathbf{D}(z, \eta)$ be the $N \times \bar{K}$-matrix with elements

$$D_{ij}(z,\eta) = E_{ij}(z,\eta) + \eta q_i^{-1} E_{ij}(z,\eta) + \alpha_i^{-1} E_{ij}(z,\eta)\mu_j(z,\eta), \tag{5.13}$$

$i \in \mathcal{N}; j \in R^-$; we then have

$$\mathbf{D}(z,\eta) = (\mathbf{I} + \eta\mathbf{q}^{-1})\mathbf{E}(z,\eta)\mathbf{I}_{N\bar{K}} + \boldsymbol{\alpha}^{-1}\mathbf{E}(z,\eta)\mathbf{I}_{N\bar{K}}\mathrm{diag}(\mu_1(z,\eta), \ldots, \mu_{\bar{K}}(z,\eta)).$$

From (5.8) it follows that for $i \in R^-$,

$$\begin{aligned} &\mathbf{H}(z, \mu_i(z,\eta), \eta)\mathbf{D}^i(z,\eta) \\ =&\boldsymbol{\alpha}^{-1}\mathbf{L}(z, \mu_i(z,\eta), \eta)\boldsymbol{\alpha}\mathbf{M}^{-1}(\mu_i(z,\eta), \eta)\boldsymbol{\alpha}^{-1}(\boldsymbol{\alpha}(\eta) + \mu_i(z,\eta)\mathbf{I})\mathbf{E}^i(z,\eta) \\ =&0. \end{aligned} \tag{5.14}$$

The matrix $\mathbf{D}(z, \eta)$ we define here corrects the corresponding matrix in [38], which wrongly does not satisfy (5.14).

Let $\mathbf{S}(z, \eta)$ be the $\bar{K} \times \bar{K}$-matrix with elements

$$S_{ij}(z,\eta) = \alpha_i^{-1} E_{ij}(z,\eta), \quad i, j \in R^-;$$

so

$$\mathbf{S}(z,\eta) = (\mathbf{I}_{\bar{K}N}\boldsymbol{\alpha}^{-1}\mathbf{I}_{N\bar{K}})(\mathbf{I}_{\bar{K}N}\mathbf{E}(z,\eta)\mathbf{I}_{N\bar{K}}). \tag{5.15}$$

We now impose the following condition.

Condition 5.3.2
For $(z,\eta) = (1,0)$ *and* $0 < |z| \leq 1, Re(\eta) > 0$, $\det \mathbf{S}(z,\eta) \neq 0$.

Define the $\bar{K} \times N$-matrix $\mathbf{C}(z,\eta)$ by

$$\mathbf{C}(z,\eta) = \mathbf{S}^{-1}(z,\eta)\mathbf{I}_{\bar{K}N}.$$

Hence, from (5.15) we have

$$\mathbf{C}(z,\eta) = (\mathbf{I}_{\bar{K}N}\mathbf{E}(z,\eta)\mathbf{I}_{N\bar{K}})^{-1}\mathbf{I}_{\bar{K}N}\boldsymbol{\alpha}.$$

Notice that the last $N-\bar{K}$ columns of $\mathbf{C}(z,\eta)$ are equal zero. Now, define the $N\times N$-matrix $\mathbf{K}(z,\phi,\eta)$ by

$$\mathbf{K}(\mathbf{z},\phi,\eta) = \mathbf{I} + \mathbf{D}(z,\eta)\text{diag}\left(\frac{1}{\phi-\mu_1(z,\eta)},\ldots,\frac{1}{\phi-\mu_{\bar{K}}(z,\eta)}\right)\mathbf{C}(z,\eta). \tag{5.16}$$

We now can prove the following factorization theorem.

Theorem 5.3.1
If Conditions 5.3.1 and 5.3.2 are satisfied, then for $(|z| < 1, Re(\eta) \geq 0)$ *or* $(|z| \leq 1, Re(\eta) > 0)$,

1. $\det \mathbf{K}(\mathbf{z},\phi,\eta) = \prod_{i=1}^{\bar{K}} \left(\frac{\phi+\alpha_i(\eta)}{\phi-\mu_i(z,\eta)}\right)$
2. *for* $Re(\phi) = 0$
   $\mathbf{H}(z,\phi,\eta) = \mathbf{H}^{+}(z,\phi,\eta)\mathbf{H}^{-}(z,\phi,\eta)$
   where
   (a) $\mathbf{H}^{-}(z,\phi,\eta) = \mathbf{K}^{-1}(z,\phi,\eta)$,
   (b) $\mathbf{H}^{+}(z,\phi,\eta) = \mathbf{H}(z,\phi,\eta)\mathbf{K}(z,\phi,\eta)$,
   (c) $\mathbf{H}^{+}(z,\phi,\eta)$ *is analytic for* $Re(\phi) > 0$, *and continuous and bounded for* $Re(\phi) \geq 0$, *and non-singular in* $Re(\phi) > 0$,
   $\mathbf{H}^{-}(z,\phi,\eta)$ *is analytic for* $Re(\phi) < 0$, *and continuous and bounded for* $Re(\phi) \leq 0$, *and non-singular in* $Re(\phi) < 0$.

Proof. Although we have different expressions for the matrices $\mathbf{D}(z,\eta)$, $\mathbf{S}(z,\eta)$ and $\mathbf{C}(z,\eta)$, the proof is essentially the same as the proof of Theorem 4.2 in [38].

Now, by multiplying both sides of (5.3) by $K(z,\phi,\eta)$ and by using (5.6) we obtain

$$\mathbf{Z}(z,\phi,\eta,v)\mathbf{H}(z,\phi,\eta)\mathbf{K}(z,\phi,\eta) = z\mathbf{Z}^0(\phi,\eta,v)\mathbf{K}(z,\phi,\eta) + \mathbf{V}(z,\phi,\eta,v)\mathbf{K}(z,\phi,\eta), \quad (5.17)$$

where from part 2.c of Theorem 5.3.1 we have that the left-hand side of (5.17) is analytic in $Re(\phi) > 0$ and bounded and continuous in $Re(\phi) \geq 0$, and the last term of the right-hand side is analytic in $Re(\phi) < 0$ and bounded and continuous in $Re(\phi) \leq 0$.

■

To obtain a standard Wiener-Hopf decomposition for (5.17) we decompose the first term of the right-hand side of (5.17), i.e. we determine matrix functions $\mathbf{K}^+$ and $\mathbf{K}^-$ such that for $Re(\phi) = 0$,

$$\mathbf{Z}^0(\phi,\eta,v)\mathbf{K}(z,\phi,\eta) = \mathbf{K}^+(z,\phi,\eta,v) + \mathbf{K}^-(z,\phi,\eta,v) \quad (5.18)$$

where

$\mathbf{K}^+(z,\phi,\eta,v)$ is analytic for $Re(\phi) > 0$, and continuous and bounded for $Re(\phi) \geq 0$,

$\mathbf{K}^-(z,\phi,\eta,v)$ is analytic for $Re(\phi) < 0$, and continuous and bounded for $Re(\phi) \leq 0$.

Lemma 5.3.2
If conditions 5.3.1 and 5.3.2 are satisfied then for $Re(\phi) \geq 0$,

$$\begin{aligned} & K^+_{ij}(z,\phi,\eta,v) \\ = \; & \delta_{ij} Z^0_i(\phi,\eta,v) + \sum_{k=1}^{\bar{K}} D_{ik}(z,\eta) \frac{Z^0_i(\phi,\eta,v) - Z^0_i(\mu_k(z,\eta),\eta,v)}{\phi - \mu_k(z,\eta)} C_{kj}(z,\eta) \end{aligned} \quad (5.19)$$

and

$$K^-_{ij}(z,\phi,\eta,v) = \sum_{k=1}^{\bar{K}} D_{ik}(z,\eta) \frac{Z^0_i(\mu_k(z,\eta),\eta,v)}{\phi - \mu_k(z,\eta)} C_{kj}(z,\eta) \quad (5.20)$$

if $(|z| < 1, Re(\eta) \geq 0)$ *or* $(|z| \leq 1, Re(\eta) > 0)$.

Proof.

- For $i = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, N$, it is clear from (5.19) and (5.20) that

$$\begin{aligned} & K^-_{ij}(z,\phi,\eta,v) + K^+_{ij}(z,\phi,\eta,v) \\ =& \delta_{ij} Z^0_i(\phi,\eta,v) + \sum_{k=1}^{\bar{K}} D_{ik}(z,\eta) \frac{Z^0_i(\phi,\eta,v)}{\phi - \mu_k(z,\eta)} C_{kj}(z,\eta) \end{aligned}$$

  so that, by using (5.16), (5.18) is satisfied.

- Since by definition $\mu_k(z,\eta)$ for $k = 1, 2, \cdots, \bar{K}$ lies in the right half-plane $Re(\phi) > 0$, $K^-_{ij}(z, \phi, \eta, v)$ is analytic for $Re(\phi) < 0$, and continuous for $Re(\phi) \leq 0$, taking into account the properties of $\mathbf{C}(z, \eta)$ and $\mathbf{D}(z, \eta)$. Furthermore, it is bounded on the left half-plane $Re(\phi) \leq 0$ since for $i = 1, 2, \cdots, \bar{K}$, the functions

$$D_{ik}(z,\eta) Z^0_i(\mu_k(z,\eta), \eta, v) C_{kj}(z, \eta)$$

and $1/(\phi - \mu_k(z, \eta))$ are bounded.

- For $i = 1, 2, \cdots, N$, $j = 1, 2, \cdots, N$ and $l = 1, 2, \cdots, \bar{K}$,

$$\begin{aligned}
&\lim_{\phi\to\mu_l(z,\eta)} (\phi - \mu_l(z,\eta)) K^+_{ij}(z, \phi, \eta, v) \\
=& \lim_{\phi\to\mu_l(z,\eta)} (\phi - \mu_l(z,\eta)) \delta_{ij} Z^0_i(\phi, \eta, v) + \Bigg[\lim_{\phi\to\mu_l(z,\eta)} (\phi - \mu_l(z,\eta)) \\
& \sum_{k=1}^{\bar{K}} D_{ik}(z,\eta) \frac{Z^0_i(\phi,\eta,v) - Z^0_i(\mu_k(z,\eta),\eta,v)}{\phi - \mu_k(z,\eta)} C_{kj}(z,\eta) \Bigg] \\
=& 0,
\end{aligned}$$

consequently, $K^+_{ij}(z, \phi, \eta, v)$ does not have any pole in the right half-plane $Re(\phi) > 0$, taking into account the properties of $\mathbf{C}(z, \eta)$ and $\mathbf{D}(z, \eta)$. We can conclude that $\mathbf{K}^+(z, \phi, \eta, v)$ is analytic for $Re(\phi) > 0$, and continuous for $Re(\phi) \geq 0$. Furthermore, it is bounded in the right half-plane $Re(\phi) \geq 0$ since for $i = 1, 2, \cdots, N$, the functions $Z^0_i(\phi, \eta, v)$, from (5.4), are bounded in the right half-plane $Re(\phi) \geq 0$.

■

Theorem 5.3.2
If conditions 5.3.1 and 5.3.2 are satisfied then for $Re(\phi) \geq 0$, *with* $(|z| < 1, Re(\eta) \geq 0)$ *or* $(|z| \leq 1, Re(\eta) > 0)$,

$$\mathbf{Z}(z,\phi,\eta,v)\mathbf{H}(z,\phi,\eta)\mathbf{K}(z,\phi,\eta) = z\mathbf{K}^+(z,\phi,\eta,v) + z\mathbf{K}^-(z,0,\eta,v) \tag{5.21}$$

Proof. From (5.17) and (5.18) we have for $Re(\phi) = 0$

$$\begin{aligned}
&\mathbf{Z}(z,\phi,\eta,v)\mathbf{H}(z,\phi,\eta)\mathbf{K}(z,\phi,\eta) - z\mathbf{K}^+(z,\phi,\eta,v) \\
=& z\mathbf{K}^-(z,\phi,\eta,v) + \mathbf{V}(z,\phi,\eta,v)\mathbf{K}(z,\phi,\eta)
\end{aligned} \tag{5.22}$$

where the left-hand side is analytic in $Re(\phi) > 0$ and continuous in $Re(\phi) \geq 0$. Furthermore, by definitions of $\mathbf{Z}(z, \phi, \eta, v)$ and $\mathbf{K}^+(z, \phi, \eta, v)$ and from part 2.c of Theorem 5.3.1, it is bounded in $Re(\phi) \geq 0$. The right-hand side is analytic in $Re(\phi) < 0$ and continuous in $Re(\phi) \leq 0$. By definitions of $\mathbf{V}(z, \phi, \eta, v)$, $\mathbf{K}(z, \phi, \eta)$ and $\mathbf{K}^-(z, \phi, \eta, v)$, it is also bounded in $Re(\phi) \leq 0$. Thus we can define an entire function which is equal to the left-hand side for $Re(\phi) \geq 0$ and equal to the right-hand side for $Re(\phi) \leq 0$. This entire function is bounded, and hence by Liouville's theorem, it is a constant. Hence, for $Re(\phi) \geq 0$,

$$\begin{aligned}
&\mathbf{Z}(z,\phi,\eta,v)\mathbf{H}(z,\phi,\eta)\mathbf{K}(z,\phi,\eta) - z\mathbf{K}^+(z,\phi,\eta,v) \\
=& \mathbf{Z}(z,0,\eta,v)\mathbf{H}(z,0,\eta)\mathbf{K}(z,0,\eta) - z\mathbf{K}^+(z,0,\eta,v).
\end{aligned} \tag{5.23}$$

Using (5.22) with $\phi = 0$ and noting that $\mathbf{V}(z, 0, \eta, v) = 0$, it follows from (5.23) that

$$\mathbf{Z}(z, \phi, \eta, v)\mathbf{H}(z, \phi, \eta)\mathbf{K}(z, \phi, \eta) = z\mathbf{K}^{+}(z, \phi, \eta, v) + z\mathbf{K}^{-}(z, 0, \eta, v).$$

This proves the theorem.

■

From (5.21) we can find an explicit expression for $\mathbf{Z}(z, \phi, \eta, v)$ once we find the explicit expressions for $\mathbf{K}(z, \phi, \eta)^{-1}$ and $\mathbf{H}^{+}(z, \phi, \eta)^{-1}$, which are given in the following lemma.

Lemma 5.3.3
For $(|z| < 1, Re(\eta) \geq 0, Re(\phi) \geq 0)$ *or* $(|z| \leq 1, Re(\eta) > 0, Re(\phi) \geq 0)$ *or* $(|z| \leq 1, Re(\eta) \geq 0, Re(\phi) > 0)$

$$\mathbf{K}^{-1}(z, \phi, \eta) = \mathbf{I} - \mathbf{D}(z, \eta)\mathbf{C}(z, \eta)\mathbf{M}^{-1}(\phi, \eta) \tag{5.24}$$

and

$$\begin{aligned}\mathbf{H}^{+}(z, \phi, \eta)^{-1} &= [\mathbf{H}(z, \phi, \eta)\mathbf{K}(z, \phi, \eta)]^{-1} \\ &= [\mathbf{M}(\phi, \eta) - \mathbf{D}(z, \eta)\mathbf{C}(z, \eta)]\, \alpha^{-1} \sum_{i=1}^{N} \frac{\mathbf{E}^{i}(z, \eta)\mathbf{E}_{i}(z, \eta)^{-1}}{(\phi - \mu_i(z, \eta))} \boldsymbol{\alpha}.\end{aligned} \tag{5.25}$$

Proof. See Appendix A.6.

The equation (5.21) and Lemma 5.3.3 will give us an explicit expression for $\mathbf{Z}(z, \phi, \eta, v)$ and we can use it to study the distribution of the buffer content.

5.4 The steady state buffer content at transition epochs

In this section we will derive the steady state distribution function of the buffer content at transition epochs of the process $\{J_t\}$ which exists for traffic intensity $\rho < 1$.

The process $\{(W_n, X_n)\}$ is regenerative where for any $i \in \mathcal{N}$ the state $(0, i)$ can be seen as the regenerative state. Since Q is indecomposable, all states of the process $\{J_t\}$ communicate with each other. It follows that the return times of the process $\{(W_n, X_n)\}$ are aperiodic so that $\lim_{n\to\infty} P\{W_n \leq x, X_n = j | X_1 = i, V_0 = v\}$ for $x \geq 0$ exists. If this limit is zero then no limiting distribution exists, otherwise (W_n, X_n) converges weakly to a stationary random vector (W, X). From (5.19) and (5.20) we see that

$$\begin{aligned}
&\lim_{z\uparrow 1}(1-z)K^+_{ij}(z,\phi,0,v)+(1-z)K^-_{ij}(z,0,0,v)\\
=&\lim_{z\uparrow 1}(1-z)\delta_{ij}Z^0_i(\phi,\eta,v)-\lim_{z\uparrow 1}(1-z)\sum_{k=1}^{\bar K}D_{ik}(z,0)\frac{Z^0_i(\mu_k(z,0),0,v)}{\mu_k(z,0)}C_{kj}(z,0)\\
&+\lim_{z\uparrow 1}(1-z)\sum_{k=1}^{\bar K}D_{ik}(z,0)\frac{Z^0_i(\phi,\eta,v)-Z^0_i(\mu_k(z,0),\eta,v)}{\phi-\mu_k(z,0)}C_{kj}(z,0) \qquad (5.26)\\
=&-D_{i1}(1,0)Z^0_i(\mu_1(1,0),0,v)\lim_{z\uparrow 1}\frac{(1-z)}{\mu_1(z,0)}C_{1j}(1,0)\\
=&-D_{i1}(1,0)\lim_{z\uparrow 1}\frac{(1-z)}{\mu_1(z,0)}C_{1j}(1,0).
\end{aligned}$$

Since $\mathbf{H}(1,0,0)=\mathbf{I}-\mathbf{G}(0,0)=\mathbf{I}-\mathbf{P}$ and $\mathbf{H}(1,0,0)\mathbf{D}^1(1,0)=0$, we may put

$$\mathbf{E}^1(1,0)=\mathbf{D}^1(1,0)=\mathbf{1}.$$

We also may put $\mathbf{E}_1^{-1}(1,0)=(\boldsymbol{\pi}\mathbf{r}\mathbf{1})^{-1}\boldsymbol{\pi}\mathbf{r}$.

From the proof of Lemma 5.3.1 we have

$$\lim_{z\uparrow 1}\frac{(1-z)}{\mu_1(z,0)}=\begin{cases}-\frac{\boldsymbol{\pi}\mathbf{r}\mathbf{1}}{\boldsymbol{\pi}\mathbf{q}\mathbf{1}} & ,\ \text{if } \boldsymbol{\pi}\mathbf{r}\mathbf{1}<0;\\ 0 & ,\ \text{if } \boldsymbol{\pi}\mathbf{r}\mathbf{1}\geq 0\end{cases} \qquad (5.27)$$

Hence, if and only if $\boldsymbol{\pi}\mathbf{r}\mathbf{1}<0$, or $\rho<1$, the vector $\{(W_n,X_n)\}$ converges weakly to (W,X). If Condition 5.3.1 and Condition 5.3.2 are satisfied we have

$$\mathbf{C}_1(1,0)=(\mathbf{I}_{\bar K N}\mathbf{E}(1,0)\mathbf{I}_{N\bar K})_1^{-1}\mathbf{I}_{\bar K N}\alpha$$

so $C_{1i}(1,0)<0, i\in R^-$; and $C_{1i}(1,0)=0, i\in R^+$. Denote $\mu_i=\mu_i(1,0), i\in\mathcal{N}$ with $\mu_1=0$ and $\mathbf{D}=\mathbf{D}(1,0), \mathbf{C}=\mathbf{C}(1,0)$ and $\mathbf{E}=\mathbf{E}(1,0)$; $\mathbf{H}(\phi)=\mathbf{H}(1,\phi,0)$, and $\tilde{\mathbf{K}}(\phi)=\mathbf{K}(1,\phi,0)$. From (5.16) it follows that

$$\tilde{\mathbf{K}}(\phi)=\mathbf{I}+\mathbf{D}\mathrm{diag}\left(0,\frac{1}{\phi-\mu_2},\ldots,\frac{1}{\phi-\mu_N}\right)\mathbf{C}.$$

Now let

$$\mathbf{H}^+(\phi)=\begin{cases}\mathbf{H}(\phi)\tilde{\mathbf{K}}(\phi)+\frac{1}{\phi}\mathbf{H}(\phi)\mathbf{D}^1\mathbf{C}_1, & \phi\neq 0,\\ \\ \mathbf{H}(0)\tilde{\mathbf{K}}(0)+\mathbf{H}'(0)\mathbf{D}^1\mathbf{C}_1, & \phi=0,\end{cases} \qquad (5.28)$$

and let $\mathbf{Z}(\phi)$ be the $N\times N-$dimensional matrix with elements

$$\begin{aligned}
Z_{ij}(\phi) &= E\left(e^{-\phi W}\mathit{1}(X=j)|X_1=i\right)\\
&= \lim_{z\uparrow 1}(1-z)Z_{ij}(z,\phi,0), j=1,2,\ldots,N.
\end{aligned}$$

From (5.21), (5.26), and (5.27) we now have for $Re(\phi) \geq 0$,

$$\begin{aligned} \mathbf{Z}(\phi)\mathbf{H}^{+}(\phi) &= \frac{\boldsymbol{\pi r 1}}{\boldsymbol{\pi q 1}}\mathbf{D}^{1}\mathbf{C}_{1} \\ &= \frac{\boldsymbol{\pi r 1}}{\boldsymbol{\pi q 1}}\mathbf{U}, \end{aligned} \tag{5.29}$$

where $\mathbf{U}$ is an $N \times N-$dimensional matrix with rows $\mathbf{C}_1$. The equation (5.29) confirms that the steady-state distribution of the buffer content does not depend on the initial condition.

Let $\mathbf{Z}_i(\phi)$ be the $N-$ dimensional row vector with components $Z_{ji}(\phi)$. From (5.29) and (5.25) we have for $Re(\phi) \geq 0$,

$$\begin{aligned} \mathbf{Z}_i(\phi) &= \frac{\boldsymbol{\pi r 1}}{\boldsymbol{\pi q 1}}\mathbf{C}_1\mathbf{H}^{+}(\phi)^{-1} \\ &= \frac{\boldsymbol{\pi r 1}}{\boldsymbol{\pi q 1}}(\mathbf{I}_{\bar{K}N}\mathbf{E}\mathbf{I}_{N\bar{K}})_1^{-1}\mathbf{I}_{\bar{K}N}\boldsymbol{\alpha}\left((\boldsymbol{\alpha}+\phi\mathbf{I})-\mathbf{DC}\right)\boldsymbol{\alpha}^{-1}\sum_{j=1}^{N}\frac{\mathbf{E}^{j}\mathbf{E}_{j}^{-1}}{\phi-\mu_j}\boldsymbol{\alpha}. \end{aligned} \tag{5.30}$$

The explicit expression for $\mathbf{Z}_i(\phi)$ can be obtain by using the following lemma.

Lemma 5.4.1
For $(|z| < 1, Re(\eta) \geq 0, Re(\phi) \geq 0)$ *or* $(|z| \leq 1, Re(\eta) > 0, Re(\phi) \geq 0)$ *or* $(|z| \leq 1, Re(\eta) \geq 0, Re(\phi) > 0)$ *and for* $i = 1, \cdots, \bar{K}$,

$$\mathbf{C}_i(z,\eta)\left[\mathbf{M}(\phi,\eta) - \mathbf{D}(z,\eta)\mathbf{C}(z,\eta)\right] = (\phi - \mu_i(z,\eta))\mathbf{C}_i(z,\eta). \tag{5.31}$$

Proof. See appendix A.7.

■

For $z = 1$ and $\eta = 0$, (5.31) yields

$$\mathbf{C}_i\left[\boldsymbol{\alpha} + \phi\mathbf{I} + \phi\mathbf{I} - \mathbf{DC}\right] = (\phi - \mu_i)\mathbf{C}_i.$$

Then by using the orthogonality property of the vectors $E^i, i \in \mathcal{N}$ and our setting

$$\mathbf{E}_1^{-1}(1,0) = (\boldsymbol{\pi}\mathbf{r}\mathbf{1})^{-1}\boldsymbol{\pi}\mathbf{r}$$

as is explained on page 114, (5.30) can be rewritten as in the following expression. For $Re(\phi) \geq 0$,

$$\begin{aligned} \mathbf{Z}_i(\phi) &= \frac{\boldsymbol{\pi r 1}}{\boldsymbol{\pi q 1}}(\phi-\mu_1)(\mathbf{I}_{\bar{K}N}\mathbf{E}\mathbf{I}_{N\bar{K}})_1^{-1}\mathbf{I}_{\bar{K}N}\sum_{j=1}^{N}\frac{\mathbf{E}^{j}\mathbf{E}_{j}^{-1}}{\phi-\mu_j}\boldsymbol{\alpha} \\ &= \frac{\boldsymbol{\pi r 1}}{\boldsymbol{\pi q 1}}\phi(\mathbf{I}_{\bar{K}N}\mathbf{E}\mathbf{I}_{N\bar{K}})_1^{-1}\mathbf{I}_{\bar{K}N}\sum_{j=1}^{N}\frac{\mathbf{E}^{j}\mathbf{E}_{j}^{-1}}{\phi-\mu_j}\boldsymbol{\alpha} \\ &= \frac{\boldsymbol{\pi r 1}}{\boldsymbol{\pi q 1}}\left(\mathbf{E}_1^{-1}\boldsymbol{\alpha} + \phi(\mathbf{I}_{\bar{K}N}\mathbf{E}\mathbf{I}_{N\bar{K}})_1^{-1}\mathbf{I}_{\bar{K}N}\sum_{j=\bar{K}+1}^{N}\frac{\mathbf{E}^{j}\mathbf{E}_{j}^{-1}}{\phi-\mu_j}\boldsymbol{\alpha}\right) \\ &= \boldsymbol{\gamma} + \frac{\boldsymbol{\pi r 1}}{\boldsymbol{\pi q 1}}\boldsymbol{\mathcal{V}}\sum_{j=\bar{K}+1}^{N}\mathbf{E}^{j}\mathbf{E}_{j}^{-1}\boldsymbol{\alpha} - \frac{\boldsymbol{\pi r 1}}{\boldsymbol{\pi q 1}}\boldsymbol{\mathcal{V}}\sum_{j=\bar{K}+1}^{N}\frac{-\mu_j}{\phi-\mu_j}\mathbf{E}^{j}\mathbf{E}_{j}^{-1}\boldsymbol{\alpha}, \end{aligned} \tag{5.32}$$

where

$$\boldsymbol{\mathcal{V}} = (\mathbf{I}_{\bar{K}N}\mathbf{E}\mathbf{I}_{N\bar{K}})_1^{-1}\mathbf{I}_{\bar{K}N}$$

and $\boldsymbol{\gamma}$ is given by (5.1). The equation (5.32) shows us that the steady-state distribution of buffer content at transition epochs is a mixture of exponentials and a concentration at 0. Let $\mathbf{F}(x)$ be the N-dimensional row vector with components

$$F_j(x) = P(W \leq x, X = j), \qquad j \in \mathcal{N}.$$

Then equation (5.32) yields

$$\mathbf{F}(x) = \boldsymbol{\gamma} + \frac{\boldsymbol{\pi r 1}}{\boldsymbol{\pi q 1}}\boldsymbol{\mathcal{V}} \sum_{j=\bar{K}+1}^{N} \mathbf{E}^j \mathbf{E}_j^{-1} e^{\mu_j x}\alpha. \tag{5.33}$$

This result can also be found in [38].

5.5 The buffer content in continuous time

In this section we consider the buffer content in continuous time. In the first part we will derive the steady state distribution function of this buffer content, and in the second part we consider the time dependent distribution.

5.5.1 The steady state buffer content in continuous time

If $J_t = j$ then

$$V_t = [W_{N_t} + r_j(t - T_{N_t})]^+$$

where N_t is the number of transitions of the process $\{J_t\}$ during $[0, t]$. Consequently, for $Re(\phi) \geq 0$,

$$\begin{aligned} &E\left(e^{-\phi V_t}\mathbf{1}(J_t = j)|X_1 = i, V_0 = v\right) \\ &= E\left(e^{-\phi[W_0 + r_j(t-T_0)]^+} \mathit{1}(T_0 \leq t < T_1, J_t = j)|X_1 = i, V_0 = v\right) \\ &+ \sum_{n=1}^{\infty} E\left(e^{-\phi[W_n + r_j(t-T_n)]^+}\mathbf{1}(T_n \leq t < T_{n+1}, J_t = j)|X_1 = i, V_0 = v\right). \end{aligned} \tag{5.34}$$

The last term on the right can be written as

$$
\begin{aligned}
&\sum_{n=1}^{\infty} E\left(e^{-\phi[W_n+r_j(t-T_n)]^+}\mathbf{1}(T_n \le t < T_{n+1}, J_t = j)|X_1 = i, V_0 = v\right) \\
=&\sum_{n=1}^{\infty}\sum_{l=1}^{N} E\left(e^{-\phi[W_n+r_j(t-T_n)]^+}\mathbf{1}(T_n \le t < T_{n+1}, X_n = l, X_{n+1} = j)|X_1 = i, V_0 = v\right) \\
=&\sum_{n=1}^{\infty}\sum_{l=1}^{N} \int_0^t P(A_{n+1} > t-u, X_{n+1} = j|X_n = l) \\
&\qquad .d_u E\left(e^{-\phi[W_n+r_j(t-T_n)]^+}\mathbf{1}(T_n \le u, X_n = l)|X_1 = i, V_0 = v\right) \\
=&\sum_{n=1}^{\infty}\sum_{l=1}^{N} \int_0^t P_{lj} e^{-q_j(t-u)} \\
&\qquad .d_u E\left(e^{-\phi[W_n+r_j(t-T_n)]^+}\mathbf{1}(T_n \le u, X_n = l)|X_1 = i, V_0 = v\right).
\end{aligned}
\tag{5.35}
$$

From the identity (see page 269 of [17])

$$
e^{-\phi x^+} = \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\frac{\phi}{\phi-\xi} e^{-\xi x}, Re(\phi) > Re(\xi) > 0,
$$

where the path of integration is a line parallel to the imaginary axis, we have

$$
\begin{aligned}
&E\left(e^{-\phi[W_n+r_j(t-T_n)]^+}\mathbf{1}(T_n \le u, X_n = l)|X_1 = i, V_0 = v\right) \\
=&\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\frac{\phi}{\phi-\xi} e^{-\xi r_j t} E\left(e^{-\xi W_n + r_j \xi T_n}\mathbf{1}(T_n \le u, X_n = l)\right. \\
&\qquad \left.|X_1 = i, V_0 = v\right), \quad Re(\phi) > Re(\xi) > 0.
\end{aligned}
\tag{5.36}
$$

Combining (5.34), (5.35), and (5.36) yields for $Re(\eta + q_j + \xi r_j) > 0$ and $Re(\phi) > Re(\xi) > 0$,

$$
\begin{aligned}
&\int_0^{\infty} e^{-\eta t} E\left(e^{-\phi V_t}\mathbf{1}(J_t = j)|X_1 = i, V_0 = v\right) dt \\
=&\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\frac{\phi}{\phi-\xi} e^{-\xi v}\int_0^{\infty} e^{-(\eta+\xi r_j)t} E\left(\mathbf{1}(t < T_1, J_t = j)|X_1 = i, V_0 = v\right) dt \\
&+\sum_{n=1}^{\infty}\sum_{l=1}^{N}\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\frac{\phi}{\phi-\xi}\int_0^{\infty} e^{-\eta t} \\
&.\int_0^t P_{lj} e^{-q_j(t-u)} e^{-\xi r_j t} d_u E\left(e^{-\xi W_n + r_j \xi T_n}\mathbf{1}(T_n \le u, X_n = l)|X_1 = i, V_0 = v\right) dt \\
=&\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\frac{\phi}{\phi-\xi} e^{-\xi v}\frac{\delta_{ij}}{\eta + \xi r_j + q_j} \\
&+\sum_{n=1}^{\infty}\sum_{l=1}^{N}\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\frac{\phi}{\phi-\xi} E\left(e^{-\xi W_n - \eta T_n}\mathbf{1}(X_n = l)|X_1 = i, V_0 = v\right) \\
&\qquad .P_{lj}(\eta + q_j + \xi r_j)^{-1}.
\end{aligned}
\tag{5.37}
$$

If $\mathbf{Z}^*(\phi, \eta, v)$ is the $N \times N$-dimensional matrix with elements

$$Z^*_{ij}(\phi, \eta, v) = \int_0^\infty e^{-\eta t} E\left(e^{-\phi V_t} \mathbf{1}(J_t = j) | X_1 = i, V_0 = v\right) dt, \tag{5.38}$$

then, for $Re(\phi) > Re(\xi) > 0, Re(\eta) > 0$, and $\max_{j \in \mathcal{N}} Re(\eta + q_j + \xi r_j) > 0$ we have

$$\begin{aligned} &\mathbf{Z}^*(\phi, \eta, v) \\ =&\frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \frac{\phi}{\phi - \xi} \left(\mathbf{Z}(1, \xi, \eta, v)\mathbf{P} + e^{-\xi v}\right) (\eta \mathbf{I} + \xi \mathbf{r} + \mathbf{q})^{-1}. \end{aligned} \tag{5.39}$$

From (5.7) and the fact that the matrix $(\eta \mathbf{I} + \xi \mathbf{r} + \mathbf{q})^{-1}$ is diagonal, we can write for $Re(\xi) > 0, Re(\eta) > 0$,

$$\mathbf{P}(\eta \mathbf{I} + \xi \mathbf{r} + \mathbf{q})^{-1} = \mathbf{G}(\xi, \eta)\mathbf{q}^{-1}.$$

Moreover, from (5.6) it follows for $Re(\xi) > 0, Re(\eta) > 0$,

$$\mathbf{P}(\eta \mathbf{I} + \xi \mathbf{r} + \mathbf{q})^{-1} = \mathbf{q}^{-1} - \mathbf{H}(1, \xi, \eta)\mathbf{q}^{-1}.$$

Multiply both sides by $\mathbf{Z}(1, \xi, \eta, v)$, then by using (5.21) we have for $Re(\xi) > 0, Re(\eta) > 0$,

$$\begin{aligned} &\mathbf{Z}(1, \xi, \eta, v)\mathbf{P}(\eta \mathbf{I} + \xi \mathbf{r} + \mathbf{q})^{-1} \\ =&\mathbf{Z}(1, \xi, \eta, v)\mathbf{q}^{-1} - \mathbf{Z}(1, \xi, \eta, v)\mathbf{H}(1, \xi, \eta)\mathbf{q}^{-1} \\ =&\mathbf{Z}(1, \xi, \eta, v)\mathbf{q}^{-1} - \mathbf{K}^+(1, \xi, \eta, v)\mathbf{K}^{-1}(1, \xi, \eta)\mathbf{q}^{-1} - \mathbf{K}^-(1, 0, \eta, v)\mathbf{K}^{-1}(1, \xi, \eta)\mathbf{q}^{-1}. \end{aligned} \tag{5.40}$$

Notice that the inverse of the matrix $\mathbf{K}(1, \xi, \eta)$ for $Re(\xi) > 0, Re(\eta) > 0$ exists due to part 1 of Theorem 5.3.1. By using (5.18) we obtain from (5.40),

$$\begin{aligned} & \mathbf{Z}(1, \xi, \eta, v)\mathbf{P}(\eta \mathbf{I} + \xi \mathbf{r} + \mathbf{q})^{-1} \\ = & \mathbf{Z}(1, \xi, \eta, v)\mathbf{q}^{-1} - \mathbf{Z}^0(\xi, \eta, v)\mathbf{K}(1, \xi, \eta)\mathbf{K}^{-1}(1, \xi, \eta)\mathbf{q}^{-1} \\ & +\mathbf{K}^-(1, \xi, \eta, v)\mathbf{K}^{-1}(1, \xi, \eta)\mathbf{q}^{-1} - \mathbf{K}^-(1, 0, \eta, v)\mathbf{K}^{-1}(1, \xi, \eta)\mathbf{q}^{-1} \\ = & \mathbf{Z}(1, \xi, \eta, v)\mathbf{q}^{-1} - \mathbf{Z}^0(\xi, \eta, v)\mathbf{q}^{-1} + \mathbf{K}^-(1, \xi, \eta, v)\mathbf{K}^{-1}(1, \xi, \eta)\mathbf{q}^{-1} \\ & -\mathbf{K}^-(1, 0, \eta, v)\mathbf{K}^{-1}(1, \xi, \eta)\mathbf{q}^{-1}, \quad Re(\xi) > 0, Re(\eta) > 0. \end{aligned}$$

Insertion into (5.39) yields for $Re(\phi) > 0$, $Re(\eta) > 0$,

$$\begin{aligned} \mathbf{Z}^*(\phi, \eta, v) = & \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \frac{\phi}{\phi - \xi} \left(\mathbf{Z}(1, \xi, \eta, v)\mathbf{q}^{-1} - \mathbf{Z}^0(\xi, \eta, v)\mathbf{q}^{-1}\right) \\ & + \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \frac{\phi}{\phi - \xi} \mathbf{K}^-(1, \xi, \eta, v)\mathbf{K}^{-1}(1, \xi, \eta)\mathbf{q}^{-1} \\ & - \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \frac{\phi}{\phi - \xi} \mathbf{K}^-(1, 0, \eta, v)\mathbf{K}^{-1}(1, \xi, \eta)\mathbf{q}^{-1} \\ & + \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \frac{\phi}{\phi - \xi} e^{-\xi v} (\eta \mathbf{I} + \xi \mathbf{r} + \mathbf{q})^{-1}. \end{aligned} \tag{5.41}$$

Since the elements of matrices $\mathbf{Z}(1, \xi, \eta, v)$ and $\mathbf{Z}^0(\xi, \eta, v)$ satisfy A^+(see page 11 for the definitions of A^+ and A^-), then the first integral in the right hand-side of (5.41) can be evaluated through the residue at $\xi = \phi$. The elements of matrices

$$\mathbf{K}^-(1, \xi, \eta, v)\mathbf{K}^{-1}(1, \xi, \eta) \text{ and } \mathbf{K}^-(1, 0, \eta, v)\mathbf{K}^{-1}(1, \xi, \eta)$$

satisfy A^-, so we can evaluate the second and the third integral through the residue at $\phi = 0$, and yields the same results. It follows that the second and the third integrals will cancel. The last integral can be evaluated by using contour integration and Cauchy's residue theorem. We recall that the jth diagonal elements of the diagonal matrix $(\eta\mathbf{I} + \xi\mathbf{r} + \mathbf{q})^{-1}$ has a pole at $\xi = -(\eta + q_j)/r_j$. For $j \in R^-$ this pole lies in the right half-plane $Re(\xi) > 0$, and for $j \in R^+$ this pole lies in the left half-plane $Re(\xi) < 0$. If we evaluate the integral

$$\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\frac{\phi}{\phi - \xi}e^{-\xi v}(\eta\mathbf{I} + \xi\mathbf{r} + \mathbf{q})^{-1}$$

through the residue in the right half-plane of the contour of integration, then we obtain for $Re(\phi) \geq 0, Re(\eta) \geq 0$,

$$\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\frac{\phi}{\phi - \xi}e^{-\xi v}(\eta\mathbf{I} + \xi\mathbf{r} + \mathbf{q})^{-1} = \mathbf{r}^{-1}\text{diag}\left(Z_1^1(\phi, \eta, v), \cdots, Z_N^1(\phi, \eta, v)\right),$$

where

$$Z_i^1(\phi, \eta, v) = \begin{cases} (\alpha_i(\eta)e^{-\phi v} + \phi e^{\alpha_i(\eta)v})/(\alpha_i(\eta)(\phi + \alpha_i(\eta))), & \text{for } i = 1, \cdots, \bar{K}, \\ \\ e^{-\phi v}/(\phi + \alpha_i(\eta)), & \text{for } i = \bar{K} + 1, \cdots, N. \end{cases}$$

Let

$$\mathbf{Z}^1(\phi, \eta, v) = \text{diag}\left(Z_1^1(\phi, \eta, v), \cdots, Z_N^1(\phi, \eta, v)\right).$$

From (5.4) it is clear that

$$\mathbf{r}^{-1}\mathbf{Z}^1(\phi, \eta, v) = \mathbf{Z}^0(\phi, \eta, v)\mathbf{q}^{-1},$$

so that if we substitute the results of integrations in (5.41) we obtain for $Re(\phi) \geq 0, Re(\eta) \geq 0$,

$$\mathbf{Z}^*(\phi, \eta, v) = \mathbf{Z}(1, \phi, \eta, v)\mathbf{q}^{-1}. \tag{5.42}$$

The process $\{(V_t, J_t), t \geq 0\}$ is regenerative, where the regeneration points are the epochs at which the process enters a state $(0, i)$ for some fixed $i \in \mathcal{N}$. Since the times between regeneration points are non-arithmetic,

$$\lim_{t\to\infty} E(exp(-\phi V_t)\mathbf{1}(J_t = i)|\mathbf{X}_1 = j, V_0 = v)$$

exists, independent of initial conditions. Denote this limit by $Z_i^*(\phi)$ and let $\mathbf{Z}^*(\phi)$ be the $N-$dimensional row vector with components $Z_i^*(\phi)$. Similar to the proof given for the process $\{(W_n, X_n)\}$, we can conclude that $\{(V_t, J_t)\}$ converges weakly to a random vector (V, J) if and only if $\boldsymbol{\pi r 1} < 0$.

To get the steady-state transform $\mathbf{Z}^*(\phi)$, in the following we apply Abel's theorem for Laplace transforms (see Appendix). From (5.21), (5.20), and (5.19), we have for $Re(\phi) \geq 0$,

$$\begin{aligned}\lim_{\eta \downarrow 0} \eta \mathbf{Z}^*(\phi, \eta, v) &= \lim_{\eta \downarrow 0} \eta \mathbf{Z}(1, \phi, \eta, v)\mathbf{q}^{-1} \\ &= \lim_{\eta \downarrow 0} \eta \left[\mathbf{K}^+(1, \phi, \eta, v) + \mathbf{K}^-(1, 0, \eta, v)\right] \left[\mathbf{H}(1, \phi, \eta) K(1, \phi, \eta)\right]^{-1} \\ &= \lim_{\eta \downarrow 0} \eta \mathbf{K}^-(1, 0, \eta, v) \left[\mathbf{H}(1, \phi, \eta) K(1, \phi, \eta)\right]^{-1}.\end{aligned} \tag{5.43}$$

From (5.25) it is clear that

$$\lim_{\eta \downarrow 0} \left[\mathbf{H}(1, \phi, \eta) K(1, \phi, \eta)\right]^{-1} = \mathbf{H}^+(\phi)^{-1},$$

where $\mathbf{H}^+(\phi)$ is defined in (5.28). To evaluate $\lim_{\eta \downarrow 0} \eta \mathbf{K}^-(1, 0, \eta, v)$ we first prove (the proof can be done in a similar way as the proof of Lemma 5.3.1)

$$\lim_{\eta \downarrow 0} \frac{\eta}{\mu_1(1, \eta)} = \begin{cases} -\boldsymbol{\pi r 1} & \text{, if } \boldsymbol{\pi r 1} < 0, \\ 0 & \text{, if } \boldsymbol{\pi r 1} \geq 0. \end{cases} \tag{5.44}$$

Then, by our setting $\mathbf{D}^1(1, 0) = \mathbf{E}^1(1, 0) = \mathbf{1}$ as explained on page 114,

$$\begin{aligned}\lim_{\eta \downarrow 0} \eta \mathbf{K}^-_{ij}(1, 0, \eta, v) &= \lim_{\eta \downarrow 0} \eta \sum_{k=1}^{\bar{K}} D_{ik}(1, \eta) \frac{Z_i^0(\mu_k(1, \eta), \eta, v)}{-\mu_k(1, \eta)} C_{kj}(1, \eta) \\ &= \lim_{\eta \downarrow 0} \frac{\eta}{-\mu_1(1, \eta)} D_{i1}(1, \eta) Z_i^0(\mu_1(1, \eta), \eta, v) C_{1j}(1, \eta) \\ &= \begin{cases} -\boldsymbol{\pi r 1} \mathbf{C}_{1j} & \text{, if } \boldsymbol{\pi r 1} < 0, \\ 0 & \text{, if } \boldsymbol{\pi r 1} \geq 0, \end{cases}\end{aligned} \tag{5.45}$$

or

$$\lim_{\eta \downarrow 0} \eta \mathbf{K}^-(1, 0, \eta, v) = \begin{cases} -\boldsymbol{\pi r 1} \mathbf{U} & \text{, if } \boldsymbol{\pi r 1} < 0, \\ \mathbf{0} & \text{, if } \boldsymbol{\pi r 1} \geq 0, \end{cases} \tag{5.46}$$

where $\mathbf{U}$ is the $N \times N$-dimensional matrix with rows $\mathbf{C}_1$. It follows that

$$\lim_{\eta \downarrow 0} \eta \mathbf{Z}^*(\phi, \eta, v) = \begin{cases} -\boldsymbol{\pi r 1} \mathbf{U} \mathbf{H}^+(\phi)^{-1} & \text{, if } \boldsymbol{\pi r 1} < 0, \\ \mathbf{0} & \text{, if } \boldsymbol{\pi r 1} \geq 0, \end{cases} \tag{5.47}$$

which confirms that $\lim_{t \to \infty} E(exp(-\phi V_t)\mathbf{1}(J_t = i) | \mathbf{X}_1 = j, V_0 = v)$ does not depend on the initial conditions. From (5.47) and (5.29) we can conclude that for $Re(\phi) \geq 0$,

$$\mathbf{Z}^*(\phi) = \boldsymbol{\pi q 1} \mathbf{Z}(\phi) \mathbf{q}^{-1}, \tag{5.48}$$

which is again the Laplace-Stieltjes transform of a mixture of exponentials and a concentration at 0.

The distribution functions

$$F_i^*(x) = P(V \le x, J = i), \quad i \in \mathcal{N},$$

can be obtained by inverting the entry in the i-th column of $\mathbf{Z}^*(\phi)$, which yields

$$F_i^*(x) = \left(\boldsymbol{\pi} + \boldsymbol{\pi r} \mathbf{1} \mathcal{V} \sum_{j=\bar{K}+1}^{N} \mathbf{E}^j \mathbf{E}_j^{-1} e^{\mu_j x} \mathbf{r}^{-1} \right)_i . \tag{5.49}$$

These results can also be found in [38].

5.5.2 Inversions for Time-dependent Buffer Content

In the previous subsection, we have derived the Laplace transform of the steady state distribution of the buffer content. An explicit expression for the distribution has been given by inverting the transform.

Now, we are interested in the distribution function of the buffer content at time $t \ge 0$ for an initial buffer content $V_0 = v$, i.e. $F_{ij}(x, t, v)$, for $i, j \in \mathcal{N}$. The double Laplace transform of these distributions is defined in (5.38), where the expressions for these transforms is given in matrix form by equation (5.42), that is for $Re(\phi) \ge 0$, $Re(\eta) > 0$,

$$\begin{aligned} \mathbf{Z}^*(\phi, \eta, v) =& \mathbf{Z}(1, \phi, \eta, v)\mathbf{q}^{-1} \\ =& (\mathbf{K}^+(1, \phi, \eta, v) + \mathbf{K}^-(1, 0, \eta, v))[\mathbf{H}(1, \phi, \eta)\mathbf{K}(1, \phi, \eta)]^{-1}\mathbf{q}^{-1}, \end{aligned} \tag{5.50}$$

where the latter expression is obtained from Theorem 5.3.2. We will show below that (5.50) holds for $Re(\phi) > 0$ and $Re(\eta) \ge 0$. We have to invert the transform in (5.50) to get the distribution functions. From (5.50) we see that the expression for $\mathbf{Z}^*(\phi, \eta, v)$ involves the matrices $\mathbf{K}^-(1, \phi, \eta, v)$ and $[\mathbf{H}(1, \phi, \eta)\mathbf{K}(1, \phi, \eta)]^{-1}$ which are rational in ϕ and the irrational diagonal matrix $\mathbf{Z}^0(\phi, \eta, v)$ with its diagonal elements that can be rewritten as

$$Z_i^0(\phi, \eta, v) = \begin{cases} (\alpha_i \alpha_i(\eta) e^{-\phi v} + \alpha_i \phi e^{\alpha_i(\eta) v})/(\alpha_i(\eta)(\phi + \alpha_i(\eta))), & \text{if } r_i < 0 \\ e^{-\phi v}/(\phi + \alpha_i(\eta)), & \text{if } r_i > 0, \end{cases}$$

so that

$$\begin{aligned} &\mathbf{Z}^0(\phi, \eta, v) \\ =& \text{diag}\left(\frac{\alpha_1(\eta) e^{-\phi v} + \phi e^{\alpha_1(\eta) v}}{\alpha_1(\eta)}, \cdots, \frac{\alpha_{\bar{K}}(\eta) e^{-\phi v} + \phi e^{\alpha_{\bar{K}}(\eta) v}}{\alpha_{\bar{K}}(\eta)}, e^{-\phi v}, \cdots, e^{-\phi v}\right) \\ &.\boldsymbol{\alpha} \mathbf{M}(\phi, \eta)^{-1} \end{aligned} \tag{5.51}$$

From (5.51) we see that the irrational property of the matrices $\mathbf{Z}^0(\phi, \eta, v)$ is caused by the exponential factors $e^{-\phi v}$. This allows us to just consider the rational parts of $\mathbf{Z}^*(\phi, \eta, v)$ in ϕ, and we can apply an analytic inversion of $\mathbf{Z}^*(\phi, \eta, v)$ with respect to ϕ, since the inversion of a term like $e^{-\phi u} \tilde{f}(\phi)$ is $f(x-u)\mathcal{H}(x-u)$, where $f(x)$ is the inverse of Laplace transform $\tilde{f}(\phi)$ and $\mathcal{H}(x)$ is the Heaviside function.

After the inversion with respect to ϕ, we get the Laplace-Stieltjes transform

$$\int_0^{\infty} e^{-\eta t} d_x F_{ij}(x,t,v)dt. \tag{5.52}$$

Since we are interested in the distribution functions, we then derive the Laplace transform

$$\xi_{ij}(x,\eta,v) = \int_0^{\infty} e^{-\eta t} F_{ij}(x,t,v)dt. \tag{5.53}$$

The transform (5.53) in general is not a rational function in η. Hence, we invert (5.53) numerically to get the desired distribution functions.

For brevity, we suppress the dependency of $\mathbf{E}(1,\eta)$ and $\bar{\mu}_i = \mu_i(1,\eta)$, $i \in \mathcal{N}$ on the variable η, and in the rest of this chapter we write $\bar{\mathbf{E}}$ and $\bar{\mu}_i$ instead of $\mathbf{E}(1,\eta)$ and

$$\bar{\mu}_i = \mu_i(1,\eta),\ i \in \mathcal{N}.$$

Let

$$\begin{aligned} &\tilde{\mathbf{Z}}^{\mathbf{0}}(\phi,\eta,\mathbf{v}) \\ =&\mathrm{diag}(\frac{\alpha_1(\eta)e^{-\phi v} + \phi e^{\alpha_1(\eta)v}}{\alpha_1(\eta)}, \cdots, \frac{\alpha_{\bar{K}}(\eta)e^{-\phi v} + \phi e^{\alpha_{\bar{K}}(\eta)v}}{\alpha_{\bar{K}}(\eta)}, e^{-\phi v}, \cdots, e^{-\phi v}). \end{aligned} \tag{5.54}$$

Lemma 5.5.1
From (5.50), Lemma 5.3.3, (5.20), (5.19), and (5.54) we obtain for $Re(\phi) \geq 0$, $Re(\eta) > 0$,

$$\begin{aligned} \mathbf{Z}^*(\phi,\eta,v) =& \tilde{\mathbf{Z}}^{\mathbf{0}}(\phi,\eta,\mathbf{v}) \sum_{l=1}^{N} \frac{\bar{\mathbf{E}}^l \bar{\mathbf{E}}_l^{-1}}{(\phi - \bar{\mu}_l)} \boldsymbol{\alpha}\mathbf{q}^{-1} \\ &- \sum_{l=1}^{\bar{K}} \frac{1}{\bar{\mu}_l} \tilde{\mathbf{Z}}^{\mathbf{0}}(\bar{\mu}_{\mathbf{l}},\eta,\mathbf{v}) \bar{\mathbf{E}}^l \mathbf{C}_l(1,\eta)\mathbf{q}^{-1} \\ &- \sum_{l=1}^{\bar{K}} \frac{1}{\bar{\mu}_l} \tilde{\mathbf{Z}}^{\mathbf{0}}(\bar{\mu}_{\mathbf{l}},\eta,\mathbf{v}) \bar{\mathbf{E}}^l \mathbf{C}_l(1,\eta) \boldsymbol{\alpha}^{-1} \sum_{l_1=1}^{N} \frac{\bar{\mu}_{l_1} \bar{\mathbf{E}}^{l_1} \bar{\mathbf{E}}_{l_1}^{-1}}{(\phi - \bar{\mu}_{l_1})} \boldsymbol{\alpha}\mathbf{q}^{-1}. \end{aligned} \tag{5.55}$$

Proof. See Appendix A.8.

■

For $k = 1, 2, \ldots, \bar{K}$, by using the orthogonality property of the vectors $\bar{\mathbf{E}}^i, i \in \mathcal{N}$, we have

$$
\begin{aligned}
&\lim_{\phi \to \bar{\mu}_k} (\phi - \bar{\mu}_k) \sum_{l=1}^{\bar{K}} \frac{1}{\bar{\mu}_l} \tilde{\mathbf{Z}}^{\mathbf{0}}(\bar{\mu}_{\mathbf{k}}, \eta, \mathbf{v}) \bar{\mathbf{E}}^l \mathbf{C}_l(1, \eta) \boldsymbol{\alpha}^{-1} \sum_{l_1=1}^{N} \frac{\bar{\mu}_{l_1} \bar{\mathbf{E}}^{l_1} \bar{\mathbf{E}}_{l_1}^{-1}}{(\phi - \bar{\mu}_{l_1})} \boldsymbol{\alpha} \mathbf{q}^{-1} \\
&= \lim_{\phi \to \bar{\mu}_k} (\phi - \bar{\mu}_k) \sum_{l=1}^{\bar{K}} \frac{1}{\bar{\mu}_l} \tilde{\mathbf{Z}}^{\mathbf{0}}(\bar{\mu}_{\mathbf{k}}, \eta, \mathbf{v}) \bar{\mathbf{E}}^l \left[(\mathbf{I}_{\bar{K}N} \bar{\mathbf{E}} \mathbf{I}_{N\bar{K}})^{-1} \right]_l \mathbf{I}_{\bar{K}N} \sum_{l_1=1}^{N} \frac{\bar{\mu}_{l_1} \bar{\mathbf{E}}^{l_1} \bar{\mathbf{E}}_{l_1}^{-1}}{(\phi - \bar{\mu}_{l_1})} \boldsymbol{\alpha} \mathbf{q}^{-1} \\
&= \tilde{\mathbf{Z}}^{\mathbf{0}}(\bar{\mu}_{\mathbf{k}}, \eta, \mathbf{v}) \bar{\mathbf{E}}^k \bar{\mathbf{E}}_k^{-1} \boldsymbol{\alpha} \mathbf{q}^{-1} \\
&\quad + \lim_{\phi \to \bar{\mu}_k} (\phi - \bar{\mu}_k) \sum_{l=1}^{\bar{K}} \frac{1}{\bar{\mu}_l} \tilde{\mathbf{Z}}^{\mathbf{0}}(\bar{\mu}_{\mathbf{l}}, \eta, \mathbf{v}) \bar{\mathbf{E}}^l \left[(\mathbf{I}_{\bar{K}N} \bar{\mathbf{E}} \mathbf{I}_{N\bar{K}})^{-1} \right]_l \mathbf{I}_{\bar{K}N} \sum_{l_1=\bar{K}+1}^{N} \frac{\bar{\mu}_{l_1} \bar{\mathbf{E}}^{l_1} \bar{\mathbf{E}}_{l_1}^{-1}}{(\phi - \bar{\mu}_{l_1})} \boldsymbol{\alpha} \mathbf{q}^{-1} \\
&= \tilde{\mathbf{Z}}^{\mathbf{0}}(\bar{\mu}_{\mathbf{k}}, \eta, \mathbf{v}) \bar{\mathbf{E}}^k \bar{\mathbf{E}}_k^{-1} \boldsymbol{\alpha} \mathbf{q}^{-1}.
\end{aligned}
$$

Now (5.55) yields for $k = 1, 2, \ldots, \bar{K}$,

$$\lim_{\phi \to \bar{\mu}_k} (\phi - \bar{\mu}_k) \mathbf{Z}^*(\phi, \eta, v) = 0,$$

or $\mathbf{Z}^*(\phi, \eta, v)$ has no pole at $\bar{\mu}_k$ for $k = 1, 2, \ldots, \bar{K}$.

Inverting (5.55) with respect to the variable ϕ, we obtain the expression for $\xi_{ij}(x, \eta, v)$ which is given in the following theorem.

Theorem 5.5.1
If conditions 5.3.1 and 5.3.2 are satisfied, then for $i \in R^-, Re(\eta) > 0$,

$$
\begin{aligned}
\xi_{ij}(x, \eta, v) = &- \frac{1}{r_j} \sum_{l=1}^{N} \frac{(\bar{\mathbf{E}}^l \bar{\mathbf{E}}_l^{-1})_{ij}}{\bar{\mu}_l} \left(1 - e^{\bar{\mu}_l (x - v)} \right) \mathcal{H}(x - v) \\
&+ \frac{1}{r_j} \sum_{l=\bar{K}+1}^{N} \frac{e^{\alpha_i(\eta) v}}{\alpha_i(\eta)} (\bar{\mathbf{E}}^l \bar{\mathbf{E}}_l^{-1})_{ij} e^{\bar{\mu}_l x} \\
&- \frac{1}{r_j} \sum_{l=1}^{\bar{K}} \frac{(\bar{\mathbf{E}}^l \bar{\mathbf{E}}_l^{-1})_{ij}}{\bar{\mu}_l} e^{\bar{\mu}_l (x - v)} \\
&- \frac{1}{r_j} \sum_{l_1=\bar{K}+1}^{N} \sum_{l=1}^{\bar{K}} \left(\frac{e^{-\bar{\mu}_l v}}{\bar{\mu}_l} + \frac{e^{\alpha_i(\eta) v}}{\alpha_i(\eta)} \right) \left(\bar{\mathbf{E}}^l \mathbf{C}_l(1, \eta) \boldsymbol{\alpha}^{-1} \bar{\mathbf{E}}^{l_1} \bar{\mathbf{E}}_{l_1}^{-1} \right)_{ij} e^{\bar{\mu}_{l_1} x},
\end{aligned}
\tag{5.56}
$$

and for $i \in R^+$,

$$\xi_{ij}(x,\eta,v) = -\frac{1}{r_j}\sum_{l=1}^{N}\frac{(\bar{\mathbf{E}}^l\bar{\mathbf{E}}_l^{-1})_{ij}}{\bar{\mu}_l}\left(1-e^{\bar{\mu}_l(x-v)}\right)\mathcal{H}(x-v)$$

$$-\frac{1}{r_j}\sum_{l=1}^{\bar{K}}\frac{(\bar{\mathbf{E}}^l\bar{\mathbf{E}}_l^{-1})_{ij}}{\bar{\mu}_l}e^{\bar{\mu}_l(x-v)} \tag{5.57}$$

$$-\frac{1}{r_j}\sum_{l_1=\bar{K}+1}^{N}\sum_{l=1}^{\bar{K}}\frac{e^{-\bar{\mu}_l v}}{\bar{\mu}_l}\left(\bar{\mathbf{E}}^l\mathbf{C}_l(1,\eta)\boldsymbol{\alpha}^{-1}\bar{\mathbf{E}}^{l_1}\bar{\mathbf{E}}_{l_1}^{-1}\right)_{ij}e^{\bar{\mu}_{l_1}x}.$$

We can check that for all $j = 1,2,\cdots,N$, if we take the limit $\eta\xi_{ij}(x,\eta,v)$ for $\eta \downarrow 0$, then we get the steady-state distribution function $F_i^*(x)$ given by (5.49).

To get the distribution functions

$$F_{ij}(x,t,v), \quad i,j \in \mathcal{N},$$

we invert the transforms (5.56) and (5.57) numerically, for which we use the algorithm in [3].

5.5.3 Relaxation time for distribution of buffer content

In this subsection we study the relaxation time, a measure of the speed of convergence of the time-dependent distribution to the steady-state distribution.

We refer to [14] for the definition of the relaxation time $T(F_{ij}(x,v))$ of the function $F_{ij}(x,v)$, as

$$T(F_{ij}(x,v)) = \inf\{T : |P(V_t \le x, J_t = j|X_1 = i, V_0 = v) - P(V \le x, J = j)| = O(e^{-t/T})\} \tag{5.58}$$

for all $x \ge 0$.

If $T^* = T(F_{ij}(x,v))$ is the relaxation time of the function $F_{ij}(x,v)$ for fixed i,j,v, then

$$P(V_t \le x, J_t = j|X_1 = i, V_0 = v) - P(V \le x, J = j) = e^{-t/T^*}g(t),$$

where $g(t) = O(e^{\epsilon t})(t \to \infty)$ for all $\epsilon > 0$, i.e. $g(t)$ is a function increasing slower than exponential. The behavior of the function

$$P(V_t \le x, J_t = j|X_1 = i, V_0 = v) - P(V \le x, J = j)$$

for $t \to \infty$ depends upon the singularities of the Laplace transform $\xi_{ij}(x,\eta,v)$ in the left half-plane $Re(\eta) < 0$ (see page 238 of Doetsch[25], page 148–156 of Schouten[41], and page 40 of Widder[43]). In general, for $t \to \infty$

$$P(V_t \le x, J_t = j|X_1 = i, V_0 = v) - P(V \le x, J = j) \approx e^{at}h(t),$$

where $h(t) = O(e^{\epsilon t})$ for all $\epsilon > 0$, and a is the real part of singular point of $\xi_{ij}(x, \eta, v)$ which is closest to the imaginary axis. This means that $T^* = -a^{-1}$.

From expressions (5.56) we see that the function $\xi_{ij}(x, \eta, v)$ for $i \in R^-$ has a pole at $\eta = -q_i$, for which $\alpha_i(\eta) = 0$, and some poles η_l, $l \in \mathcal{N}$, for which $\mu_l(1, \eta_l) = 0$. Moreover, from (5.57), for $i \in R^+$, the poles are η_l, $l \in \mathcal{N}$, for which $\mu_l(1, \eta_l) = 0$. It is readily verified from the definition of $L(z, \phi, \eta)$ that in this case η_l for $l \in \mathcal{N}$ are exactly the eigenvalues of the matrix $\mathbf{Q}$.

The other possible singular points of $\xi_{ij}(x, \eta, v)$ are the branch points of $\mu_l(1, \eta)$. If $\bar{\eta}$ is a branch point of $\xi_{ij}(x, \eta, v)$, then from (5.12),

$$\det(\mu(1, \bar{\eta})\mathbf{I} + \bar{\eta}\mathbf{r}^{-1} - \mathbf{r}^{-1}\mathbf{Q}) = 0, \tag{5.59}$$

and

$$d_\mu(\det(\mu(1, \bar{\eta})\mathbf{I} + \bar{\eta}\mathbf{r}^{-1} - \mathbf{r}^{-1}\mathbf{Q})) = 0. \tag{5.60}$$

For some values of $r_i, i \in \mathcal{N}$, the singular point of $\xi_{ij}(x, \eta, v)$ which is closest to the imaginary axis could be a branch point of $\mu_l(1, \eta), l \in \mathcal{N}$, and for other values of $r_i, i \in \mathcal{N}$, such a singular point could be the pole $\eta = -q_i$. This means that the relaxation time depends on the matrix $\mathbf{Q}$ and indeed also on the net input rates r_i. This shows that the conjecture in [42], which states that the relaxation time depends only on the generator $\mathbf{Q}$, is false. For an illustration, consider the following example.

Example 5.5.1

Consider a system in which the generator of the underlying Markov chain is

$$\mathbf{Q} = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 5 & -5 \end{pmatrix},$$

and the maximal output rate $c = 1$. The poles of the function $\xi_{ij}(x, \eta, v)$, which are also the eigenvalues of the generator $\mathbf{Q}$, are $\eta = -q_1 = -4$ and $\eta = -9$, the latter is the only non-zero eigenvelue of $\mathbf{Q}$. The system of equations (5.59) and (5.60) gives us the branch points

$$\bar{\eta}_1 = \frac{-r_1 q_2 + r_2 q_1 + 2\sqrt{-r_1 r_2 q_1 q_2}}{r_1 - r_2}$$

and

$$\bar{\eta}_2 = \frac{-r_1 q_2 + r_2 q_1 - 2\sqrt{-r_1 r_2 q_1 q_2}}{r_1 - r_2}.$$

For $r_1 = -1.5, \quad r_2 = 0.975$, these branch points are $\bar{\eta}_1 = -8.976425$ and $\bar{\eta}_2 = -0.2357$. The singular point closest to the imaginary axis is the branch point $\eta = -0.2357$, so that the relaxation time is $\frac{1}{0.2357}$.

One could think that the dependence of the relaxation time on the matrix $\mathbf{Q}$ and the input rates r_i can be converted to solely a dependence on the traffic intensity ρ. The examples in section 5.6 show that for fixed $\mathbf{Q}$, the relaxation time is decreasing when the traffic intensity is decreasing. It turns out that this dependence can not be interpreted as a simple dependence in the sense that the traffic intensity is the only variable that determines the value of the relaxation time, since for the same value of the traffic intensity the type of the singular point can be different and yields different relaxation times as shown in the following example.

Example 5.5.2

We consider two systems in which $\mathcal{N} = 2$ and the generator of the underlying Markov chain is

$$\mathbf{Q} = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix},$$

where the values of q_1 and q_2 for the systems are given in the table below. Both systems have the same traffic intensity, $\rho = 0.2857$.

q_1	q_2	r_1	r_2	ρ	type of the singlular point closest to the imaginary axis	relaxation time
1.0	6.0	-1.0	1.0	0.2857	pole	1.0
0.5	1.0555	-2.0	2.0	0.2857	branch point	19.498

5.6 Algorithm and numerical results

In this section we give some examples of the probability distribution of the buffer content in continuous time for the model in which the underlying Markov Chain is

$$\mathbf{Q} = \begin{pmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -5 & 2 & 0 & 2 \\ 3 & 3 & -9 & 1 & 2 \\ 1 & 2 & 1 & -8 & 4 \\ 3 & 2 & 1 & 0 & -6 \end{pmatrix}.$$

The linear system for the stationary probabilities $\boldsymbol{\pi}\mathbf{Q} = 0, \quad \boldsymbol{\pi}\mathbf{1} = 1$ yields

$$\begin{aligned} \pi_1 &= 0.399, \\ \pi_2 &= 0.255, \\ \pi_3 &= 0.126, \\ \pi_4 &= 0.058, \\ \pi_5 &= 0.222. \end{aligned}$$

We choose the input rates c_i, $i \in \mathcal{N}$ and the maximal output rate c such that condition

$$\sum_{i=1}^{N} \pi_i r_i = \sum_{i=1}^{N} \pi_i (c_i - c) < 0$$

is satisfied. The traffic intensity is given by $\rho = \sum_{i=1}^{N} \pi_i c_i / c$.

5.6.1 Algorithm

The steady-state distribution function of the buffer content is given explicitly by (5.49), and the time-dependent distribution function of the buffer content

$$F_{ij}(x,t,v) = P(V_t \le x, J_t = j | X_1 = i, V_0 = v)$$

for $i \in R^-$, can be obtained by inverting (5.56) and (5.57)numerically.

In evaluating the functions(5.49), (5.56), and (5.57) in some points of z and η, first we need to determine the matrix $\mathbf{E}(z,\eta)$ since all the functions mentioned above contain the term $\mathbf{E}^i(z,\eta)\mathbf{E}_i(z,\eta)^{-1}$ for some $i = 1, 2, \cdots, N$. As discussed on page 109, $\mathbf{E}^i(z,\eta)$, the ith column of matrix $\mathbf{E}(z,\eta)$, is a nonzero column vector satisfying

$$\mathbf{L}(z, \mu_i(z,\eta), \eta)\mathbf{E}^i(z,\eta) = 0,$$

or, from (5.9), $\mu_i(z,\eta)$ and $\mathbf{E}^i(z,\eta)$ is an eigen vector of the matrix $-\boldsymbol{\alpha}(\eta) + z\boldsymbol{\alpha} + z\mathbf{r}^{-1}\mathbf{Q}$ which is associated to the eigen value $\mu_i(z,\eta)$. If condition 5.3.1 is fulfilled, the eigen space of the eigen value $\mu_i(z,\eta)$ for $i = 1, 2, \cdots, N$, has dimension one. This means that we can choose any eigen vector associated with the eigen value $\mu_i(z,\eta)$ for the column vector $\mathbf{E}^i(z,\eta)$, since the other eigen vectors are just the multiplication of $\mathbf{E}^i(z,\eta)$ with some positive scalar. Choosing another eigen vector for $\mathbf{E}^i(z,\eta)$ will give us the same value of $\mathbf{E}^i(z,\eta)\mathbf{E}_i(z,\eta)^{-1}$.

The approximation value of $F_{ij}(x,t,v)$ in [3] is given by

$$F_{ij}(x,t,v) \approx \frac{e^{A/2}}{t} \sum_{k=0}^{m} \binom{m}{k} 2^{-m} S_{n+k}(t), \tag{5.61}$$

where

$$S_n(t) = \sum_{k=0}^{n} (-1)^k a_k(t), \tag{5.62}$$

and

$$a_0(t) = \xi_{ij}(x, A/2t, v)/2 \tag{5.63}$$

$$a_k(t) = Re(\xi_{ij}(x, (A + 2k\pi i)/2t, v)), \quad k \ge 1. \tag{5.64}$$

A, m, and n are the parameters to control the error bound. The setting of these parameters values is discussed in [3]. As an illustration, to set the error bound of order 10^{-7} we can choose $A = 19.1, m = 11$, and $n = 15$.

5.6.2 Results

We give some numerical inversion results for the time-dependent distribution function

$$F_{ij}(x,t,v) = P(V_t \le x, J_t = j | X_1 = i, v = v_0)$$

for the model described in the beginning of this section. The function values are calculated from the approximation relation (5.61) with parameter values $A = 19.1, m = 11$, and $n = 15$. We also calculate the relaxation time, which uses the analysis as discussed in the subsection 5.5.3. Figures 5.1 to 5.4 show the time-dependent distribution function $F_{23}(x, t, 0.5)$ for some values of the traffic intensity. We can see that the relaxation time is increasing as the traffic intensity is increasing. This means that the speed of convergence is decreasing as the traffic intensity is increasing.

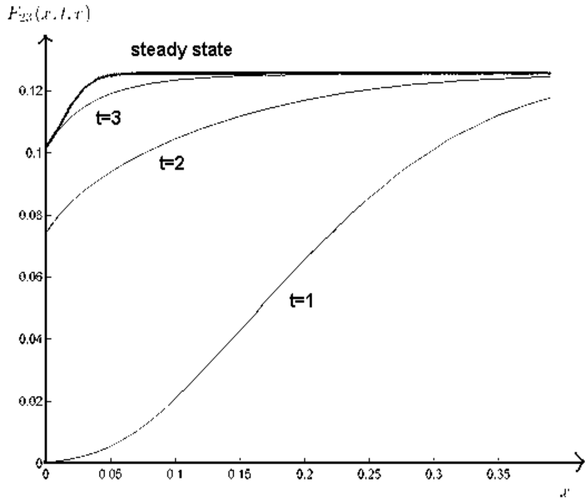

Figure 5.1: $r(1) = -0.35, r(2) = -0.25, r(3) = -0.3, r(4) = 0.3, r(5) = 0.1$, $v = 0.5$, $\rho = 0.097526$, the relaxation time $= \frac{1}{0.5} = 2.0$

Figures 5.5 to 5.7 show the time-dependent distribution function $F_{12}(x, t, v)$ for some values of the initial buffer content v. The relaxation time for all distribution functions in figures 5.5 to 5.7 is the same, i.e. 17.857. But figures 5.5 to 5.7 show us that the speed of convergence also depends on the initial buffer content.

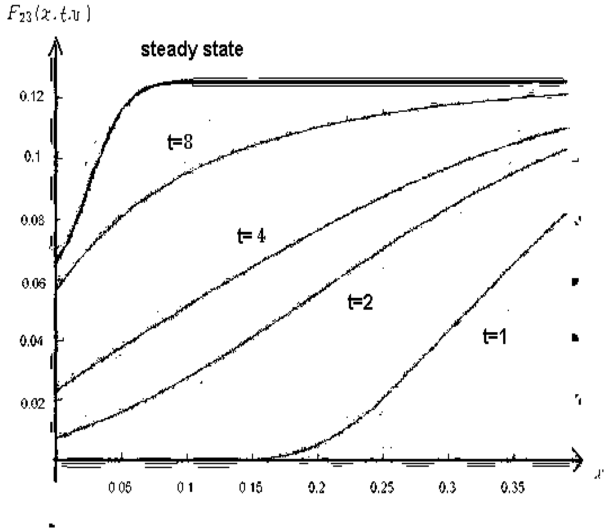

Figure 5.2: $r(1) = -0.25, r(2) = -0.15, r(3) = -0.2, r(4) = 0.4, r(5) = 0.2$, $v = 0.5$, $\rho = 0.195052$, the relaxation time $= \frac{1}{0.26} = 3.846$

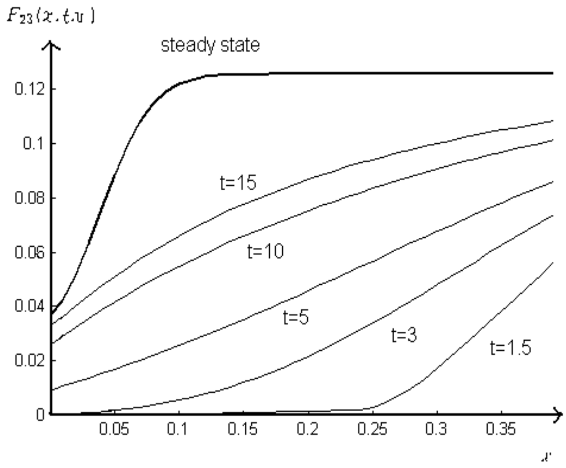

Figure 5.3: $r(1) = -0.20, r(2) = -0.125, r(3) = -0.15, r(4) = 0.45, r(5) = 0.25$, $v = 0.5$, $\rho = 0.2625$, the relaxation time $= \frac{1}{0.056} = 17.857$

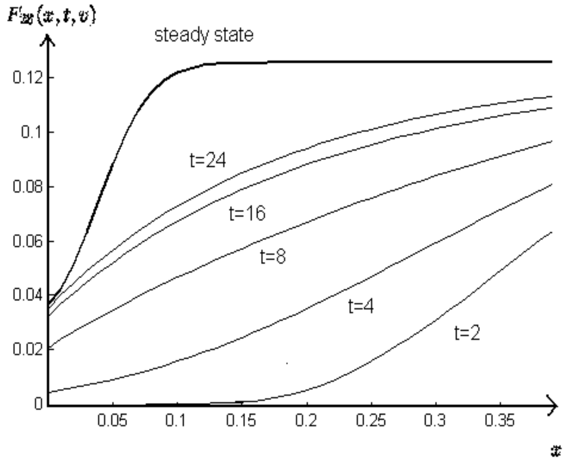

Figure 5.4: $r(1) = -0.198, r(2) = -0.098, r(3) = -0.148, r(4) = 0.452, r(5) = 0.252$, $v = 0.5$, $\rho = 0.9406358$, the relaxation time $= \frac{1}{0.035} = 28.571$

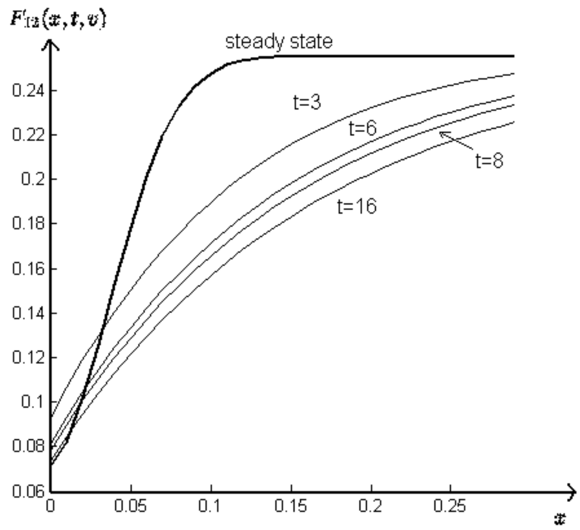

Figure 5.5: $P(V_t \leq x, J_t = 2|X_1 = 1, V_0 = 0), \rho = 0.2625, r(1) = -0.20, r(2) = -0.125, r(3) = -0.15, r(4) = 0.45, r(5) = 0.25$, the relaxation time $= \frac{1}{0.056} = 17.857$

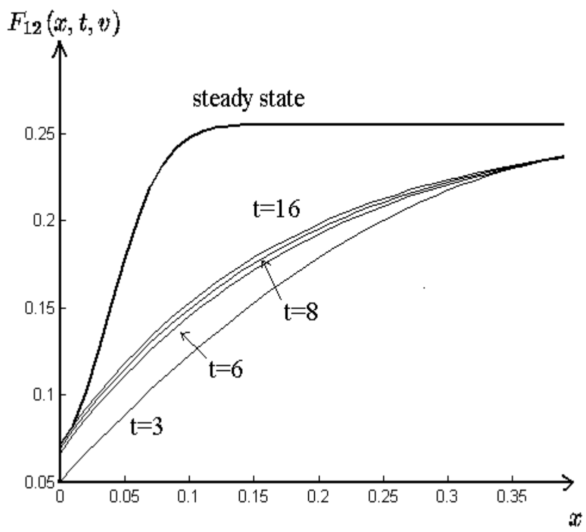

Figure 5.6: $P(V_t \leq x, J_t = 2|X_1 = 1, V_0 = 0.255208333)$, $\quad \rho = 0.2625\, r(1) = -0.20, r(2) = -0.125, r(3) = -0.15, r(4) = 0.45, r(5) = 025$, the relaxation time $= \frac{1}{0.056} = 17.857$

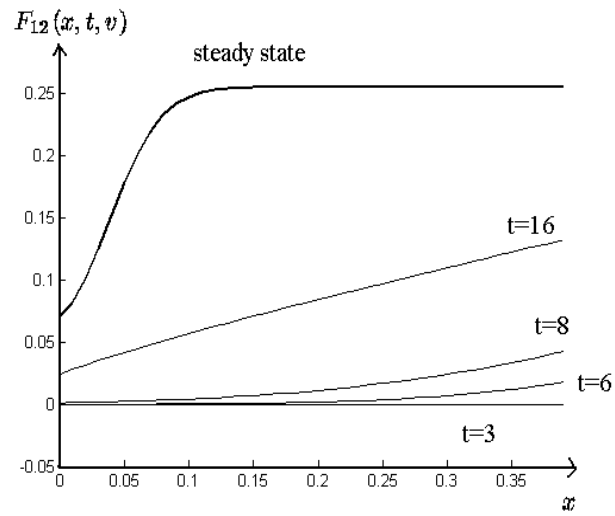

Figure 5.7: $P(V_t \leq x, J_t = 2|X_1 = 1, V_0 = 1.0)$, $\quad \rho = 0.2625\ r(1) = -0.20, r(2) = -0.125, r(3) = -0.15, r(4) = 0.45, r(5) = 025$, the relaxation time $= \frac{1}{0.056} = 17.857$

Chapter 6

Semi - Markovian Fluid Flow Model

6.1 Introduction

In this chapter we study a generalization of the model considered in chapter 5. Let $\{(A_n, X_n), n \geq 0\}$ be a Markov renewal sequence with the property that for $n = 1, 2, \cdots$,

$$\begin{aligned} &P(A_{n+1} \leq x, X_{n+1} = j | A_1, \cdots, A_n, X_1, \cdots, X_{n-1}, X_n = i) \\ =&P(A_{n+1} \leq x, X_{n+1} = j | X_n = i), \end{aligned}$$

in which the latter conditional probability does not depend on n and will be denoted as $H_{ij}(x)$. Let $T_n = \sum_{i=1}^{n} A_i$, for $n = 1, 2, \cdots$, $T_0 = 0$, and for $t \geq 0$, let

$$N_t = \sup\{n | T_n \leq t\}.$$

Now, consider the semi-Markov process $\{J_t, t \geq 0\}$ where

$$J_t = X_{N_t+1}.$$

We see that at T_n the process $\{J_t, t \geq 0\}$ jumps from one state to the next state and $X_n = J_{T_n^-}$.

The structure of the fluid flow model we study in this chapter is similar to the structure of the model in chapter 5. Let $\mathcal{N} = \{1, 2, \ldots, N\}$ be the state space of the Markov chain $\{X_n, n \geq 0\}$. We assume that this process is irreducible and aperiodic. The slope $\{a_t\}$ of the input process is constant between transitions of $\{J_t\}$ and is equal to c_i when $\{J_t\}$ is in state i. The input flows into an infinite buffer that has maximal output rate c, and initially has a content v. It follows that the rate of the net input process is also constant between transitions of $\{J_t\}$, and is equal to $r_i = c_i - c$. We define the buffer content at time t as V_t, with the assumption that $V_0 = v > 0$. Let $W_n = V_{T_n}$, $n = 0, 1, \ldots$. It is clear that $W_0 = v$.

Let H_j, $j \in \mathcal{N}$ be the time the process $\{J_t\}$ spends in state j before making a transition into a different state. In this chapter we assume that for $j \in \mathcal{N}$, H_j is hyper-exponentially distributed or hypo-exponentially distributed(see Riska[39]). In other words, for $j \in \mathcal{N}$, H_j is a mixture of m exponential distributions or the distribution of the sum of m independent exponentially distributed random veriables, $m \geq 2$. Let $\mathcal{M} = \{1, 2, \cdots, m\}$. It follows that for $i, j \in \mathcal{N}$, $n = 1, 2, \cdots$, the Laplace-Stieltjes transform

$$\begin{aligned}\mathcal{H}_{ij}(\phi) &= \int_0^\infty e^{-\phi x} dH_{ij}(x) \\ &= \int_0^\infty e^{-\phi x} dP\{A_{n+1} \le x | X_n = i, X_{n+1} = j\} P\{X_{n+1} = j | X_n = i\},\end{aligned}$$

can be expressed as

$$\begin{aligned}\mathcal{H}_{ij}(\phi) &= \mathbf{P}_{ij} \frac{h_j(\phi)}{\prod_{k=1}^m (\phi + \mu_{jk})} \\ &= \mathbf{P}_{ij} \frac{\sum_{k=1}^m a_{jk}\mu_{jk} \prod_{l=1, l\neq k}^m (\phi + \mu_{jl})}{\prod_{k=1}^m (\phi + \mu_{jk})}, \quad j \neq i,\end{aligned} \tag{6.1}$$

where for $i, j \in \mathcal{N}$, $k \in \mathcal{M}$, the constants P_{ij}, μ_{jk} and a_{jk} are described as follows.

- $\mathbf{P}_{ij}$ is the transition probability $\mathbf{P}_{ij} = P\{X_{n+1} = j | X_n = i\}$,
- $\mu_{jk} > 0$ are assumed to be distinct,
  - **Case 1** For $j \in \mathcal{N}$, H_j is hyper-exponentially distributed. Then the constants $a_{jk} > 0$ do not have to be dependent on μ_{jk}, and satisfy $\sum_{k=1}^m a_{jk} = 1$.

    Case 2 For $j \in \mathcal{N}$, H_j is hypo-exponentially distributed. Then

    $$a_{jk} = \prod_{l=1, l\neq k}^{m} \frac{\mu_{jl}}{(-\mu_{jk} + \mu_{jl})}. \tag{6.2}$$

    Notice that for this case, the function $h_j(\phi)$ can be written in the simpler form

    $$h_j(\phi) = \prod_{k=1}^{m} \mu_{jk}. \tag{6.3}$$

Our assumption in (6.1) is a generalization of the corresponding Laplace-Stieltjes transform in chapter 5 which has the form

$$\mathcal{H}_{ij}(\phi) = \begin{cases} \frac{Q_{ij}}{q_i} \frac{q_j}{\phi + q_j} & , i \neq j, \\ 0 & , i = j, \end{cases}$$

where Q_{ij} is the (i, j)th element of the infinitesimal generator of the process $\{J_t, t \ge 0\}$, and $q_i = -Q_{ii}$.

With the assumption on $\mathcal{H}_{ij}(\phi)$ above, the symbol of Wiener-Hopf-type equations is still a rational matrix in ϕ, and each element of this matrix has only simple poles. With this property, this matrix can be factorized by the Wiener-Hopf factorization technique as we apply in chapter 5. For more general models in which the times between transitions are not hyper-exponential or hypo-exponential, but $\mathcal{H}_{ij}(\phi)$ is still a rational function of ϕ, the

Wiener-Hopf factorization technique still can be used to solve the problem. If the symbol of Wiener-Hopf-type equations has some poles of order more than one, the construction of factors of the symbol will be different from what we did in section 5.3 and it needs a more complicated analysis.

We are interested in the probability distribution of the buffer content in steady state as well as in the time-dependent case, in which the distribution functions at time $t \geq 0$ are denoted by

$$F_{ij}(x,t,v) = P(V_t \leq x, J_t = j | X_1 = i, V_0 = v), \quad i,j \in \mathcal{N}.$$

Although most fluid flow models studied so far have a Markovian underlying process, Kulkarni[32] has suggested an analysis of the semi-Markovian fluid flow models. It is assumed that the analysis is going to be rather hard, and indeed, most papers that study the semi-Markovian fluid flow models, i.e. Kella and Whitt[29], Gautam *et al.*[27], Asmussen[8], Boxma *et al.*[15], and Latouche and Takine[34], focus the analysis on special cases of the models.

The steady-state distribution of the buffer content of the present semi-Markovian fluid flow model is studied in [27]. The upper and lower bounds for the steady-state distribution are derived, and discussion on some examples and applications in telecommunication networks can be found in this paper. In [34], the study is focused on the semi-Markovian fluid flow model in which the intervals during which the input rate is negative(positive) have an exponential distribution. The structure of the steady-state buffer content distribution is studied by applying the Markov-renewal approach developed earlier in the context of quasi-birth-and-death processes and of Markovian fluid queues. In [15], a model is studied in which the underlying semi-Markov process has three states where at least one of the periods in a state has a general distribution and the others have exponential distributions. The distribution of the buffer content, the distribution of the busy period and the distribution of the maximal buffer content during a busy period, in steady-state, are obtained by establishing relations between the fluid flow models and ordinary queues with instantaneous input, and by using level crossing theory. The approach is an extension of the one in [29], which only uses the relations between the fluid flow models and ordinary queues with instantaneous input. A more general model, in which the period the underlying process $\{J_i\}$ spends in a state has a general phase-type distribution, is studied in [8]. In the latter paper the steady-state distribution of the buffer content is derived.

As in chapter 5, in section 6.2 we consider the process $\{(W_n, T_n, X_n)\}$ and derive Wiener-Hopf type equations for the transform of the joint distribution of $\{(W_n, T_n, X_n)\}$. The system of equations we obtain for the present model is similar to the system in chapter 5. The only difference is that the expression for its symbol is a more general rational function, since now we are dealing with a more general transform $H_{ij}(\phi)$. In section 6.3 we solve this system of equations with Wiener-Hopf factorization, which based on the minimal representation of polynomial matrices (see Gohberg *et al*[28] for the explanation of the minimal representation) . By using the characteristic of the minimal representation, we obtain an explicit expression for the transform of $\{(W_n, T_n, X_n)\}$. The double transform of the buffer content in continuous time can be derived from the transform of $\{(W_n, T_n, X_n)\}$, and this double transform with respect to one transform variable is a closed form, so that we can invert it analytically to obtain the Laplace-Stieljes transform of the buffer content in continuous time, as we do in section 6.4. To obtain the time-dependent distribution of the buffer content in continuous time, we invert its Laplace-Stieltjes transform numerically. The steady-state distribution can be obtained by applying Abel's limit theorem to the transform. We show that the steady-state distribution function has a similar structure as for the Markovian case we studied in chapter 5.

We implement the numerical inversion algorithm given in [3] to get the time-dependent distribution functions, and the results can be found in section 6.6. The behavior of the time-dependent distributions we obtained from the numerical inversion confirms our conjecture in chapter 5, that the speed of convergence of the time-dependent distributions to the steady-state distribution depends on the transition matrix $\mathbf{P}$, the distribution of H_j, $j \in \mathcal{N}$, the initial buffer content v, and the net input rates r_i, $i \in \mathcal{N}$.

We will use the following notations: $x^+ = \max(0, x)$, and $x^- = \min(0, x)$. $\mathit{1}$ is the indicator function, $\mathbf{1}$ is the N-dimensional column vector with all components equal to 1, $\mathbf{1}_i$ is the $\bar{K}$-dimensional column vector with i-th component 1 and all other component equal to 0, where $\bar{K}$ is an integer defined in section 5.2. $\mathbf{I}$ is the identity matrix, $\mathbf{I}_{kl}$ is the $k \times l$-matrix with elements δ_{ij}, where δ_{ij} is Kronecker's delta, i.e., $\delta_{ij} = 0$, for $i \neq j$, and $\delta_{jj} = 1$. If $\mathbf{A}$ is an $N \times N$-dimensional matrix, the i-th column of $\mathbf{A}$ is denoted by $\mathbf{A}^i$, and the i-th row of $\mathbf{A}$ is denoted by $\mathbf{A}_i$. If $\mathbf{A}(i), i = 1, \cdots, m$ are $N \times N$-dimensional matrices with elements $\mathbf{A}(i)_{jk}$, we denote by $\sum_{i=1}^{m} \mathbf{A}(i)$ the $N \times N$-dimensional matrix with elements $\sum_{i=1}^{m} \mathbf{A}(i)_{jk}$, and we denote by $\prod_{i=1}^{m} \mathbf{A}(i)$ the $N \times N$-dimensional matrix where the (j, k)th element is given by the multiplication of $\mathbf{A}(i)_{jk}$ for $i = 1, 2, \cdots, m$.

6.2 System of Wiener-Hopf type equations

Let $\mathbf{P}$ be the transition probability matrix of the Markov chain$\{X_n\}$ with elements P_{ij}. We assume that the Markov chain $\{X_n\}$ is irreducible and positive recurrent. The stationary probabilities $\lim_{n\to\infty} P(X_n = i)$ are denoted by p_i, $i \in \mathcal{N}$, and $\mathbf{p}$ denotes the N-dimensional row vector with components p_i. From (6.1) we see that the times between transitions of $\{J_t\}$ are non-arithmetic so that $\lim_{t\to\infty} P(J_t = i)$, $i \in \mathcal{N}$ exists. The stationary probabilities $\lim_{t\to\infty} P(J_t = i)$ are denoted by π_i, $i \in \mathcal{N}$ and $\boldsymbol{\pi}$ denotes the N-dimensional row vector with components π_i.

We assume that $\sum_{i=1}^{N} \pi_i r_i < 0$ to ensure stability. The traffic intensity ρ, i.e. the ratio of the average input rate and the maximal output rate , is $\rho = \sum_{i=1}^{N} \pi_i c_i / c$.

We assume that for $i \in \mathcal{N}$, $c_i \neq c$ so that $r_i \neq 0$ for $i \in \mathcal{N}$. Let

$$R^- = \{i | r_i < 0, i = 1, \ldots, N\} \text{ and } R^+ = \{i | r_i > 0, i = 1, \ldots, N\}.$$

Let $|R^-| = \bar{K}$. This implies that $|R^+| = N - \bar{K}$. Let $\mathbf{r} = \mathrm{diag}(r_1, \ldots, r_N)$. Without loss of generality, suppose that $R^- = \{1, 2, \ldots, \bar{K}\}$.

Define for $Re(\eta) \geq 0, Re(\phi) \geq 0, v \geq 0$,

$$Z_i^0(\phi, \eta, v) = E\left(e^{-\phi W_1 - \eta T_1} \mathit{1}(X_1 = i) | X_1 = i, V_0 = v\right), \tag{6.4}$$

and define for $(|z| < 1, Re(\eta) \geq 0, Re(\phi) \geq 0)$ or $(|z| \leq 1, Re(\eta) > 0, Re(\phi) \geq 0)$ or $(|z| \leq 1, Re(\eta) \geq 0, Re(\phi) > 0)$

$$Z_{ij}(z, \phi, \eta, v) = \sum_{n=1}^{\infty} z^n E\left(e^{-\phi W_n - \eta T_n} \mathit{1}(X_n = j) | X_1 = i, V_0 = v\right).$$

Define for $Re(\phi) \leq 0$,

$$\begin{aligned} V_{ij}(z, \phi, \eta, v) &= \sum_{n=1}^{\infty} z^{n+1} E\Big(\Big(1 - e^{-\phi[W_n + R_{n+1}A_{n+1}]^-}\Big) \\ &\qquad e^{-\eta(T_n + A_{n+1})} \mathit{1}(X_n = j) | X_1 = i, V_0 = v\Big), \end{aligned}$$

and for $Re(\phi) = 0$,

$$G_{ij}(\phi, \eta) = E\left(e^{-(r_j\phi + \eta)A_{n+1}} \mathit{1}(X_{n+1} = j) | X_n = i\right).$$

Let $\mathbf{Z}(z, \phi, \eta, v), \mathbf{V}(z, \phi, \eta, v)$ and $\mathbf{G}(\phi, \eta)$ be $N \times N$-matrices with elements $Z_{ij}(z, \phi, \eta, v)$, $V_{ij}(z, \phi, \eta, v)$ and $G_{ij}(\phi, \eta)$ respectively.

The assumption given in equation (6.1) yields

$$G_{ij}(\phi, \eta) = P_{ij} \frac{h_j(\phi r_j + \eta)}{\prod_{k=1}^{m} (\phi r_j + \eta + \mu_{jk})} = P_{ij} \sum_{k=1}^{m} \frac{a_{jk}\mu_{jk}}{(\phi r_j + \eta + \mu_{jk})},$$

where the constants a_{jk} satisfy the conditions explained on page 134. We then obtain the following system of Wiener-Hopf-type equations in matrix notation.

Theorem 6.2.1
For $Re(\phi) = 0$ and $(|z| \leq 1, Re(\eta) > 0)$ or $(|z| < 1, Re(\eta) \geq 0)$ we have

$$\mathbf{Z}(z, \phi, \eta, v)(\mathbf{I} - z\mathbf{G}(\phi, \eta)) = z\mathbf{Z}^0(\phi, \eta, v) + \mathbf{V}(z, \phi, \eta, v), \tag{6.5}$$

where

$$\mathbf{Z}^0(\phi, \eta, v) = diag(Z_1^0(\phi, \eta, v), Z_2^0(\phi, \eta, v), \ldots, Z_N^0(\phi, \eta, v)),$$

with

$$Z_i^0(\phi, \eta, v) = \begin{cases} e^{-\phi v} \dfrac{h_i(\phi r_i + \eta)}{\prod_{k=1}^{m} (\phi r_i + \eta + \mu_{ik})} & , \text{ if } r_i > 0, \\ \displaystyle\sum_{k=1}^{m} a_{ik}\mu_{ik} \left[\frac{e^{-\phi v} - e^{(\eta + \mu_{ik})v/r_i}}{(\phi r_i + \eta + \mu_{ik})} + \frac{e^{(\eta + \mu_{ik})v/r_i}}{(\eta + \mu_{ik})} \right] & , \text{ if } r_i < 0. \end{cases} \tag{6.6}$$

Proof. For the proof of (6.5), see the proof of Theorem 5.2.1. To get the expression for $Z_i^0(\phi, \eta, v)$, we recall that

$$\begin{aligned} Z_i^0(\phi, \eta, v) &= E\left(e^{-\phi W_1 - \eta T_1} \mathbf{1}(X_1 = i) | X_1 = i, V_0 = v\right) \\ &= \int_0^{\infty} E\left(e^{-\phi[v + r_i T_1]^+ - \eta T_1} | T_1 = u\right) dP\{T_1 \leq u | X_1 = i\}. \end{aligned}$$

Since $T_1 = A_1$, it follows from our assumption in (6.1) that for $r_i < 0$ and $v \geq 0$,

$$\begin{aligned} Z_i^0(\phi,\eta,v) &= \int_0^{-v/r_i} E\left(e^{-\phi v-(\phi r_i+\eta)T_1}|T_1=u\right) dP\{A_1 \leq u|X_1=i\} \\ &+ \int_{-v/r_i}^{\infty} E\left(e^{-\eta T_1}|T_1=u\right) dP\{A_1 \leq u|X_1=i\} \\ &= \sum_{k=1}^{m} a_{ik}\mu_{ik}\frac{\left[e^{-\phi v}-e^{(\eta+\mu_{ik})v/r_i}\right]}{(\phi r_i+\eta+\mu_{ik})} \\ &+ \sum_{k=1}^{m} a_{ik}\mu_{ik}\frac{e^{(\eta+\mu_{ik})v/r_i}}{(\eta+\mu_{ik})}, \end{aligned}$$

and for $r_i > 0$ and $v \geq 0$,

$$\begin{aligned} Z_i^0(\phi,\eta,v) &= \int_0^{\infty} E\left(e^{-\phi v-(\phi r_i+\eta)T_1}|T_1=u\right) dP\{A_1 \leq u|X_1=i\} \\ &= e^{-\phi v}\frac{h_i(\phi r_i+\eta)}{\prod_{k=1}^{m}(\phi r_i+\eta+\mu_{ik})}. \end{aligned}$$

■

The system in Theorem 6.2.1 will be solved by the Wiener-Hopf factorization method.

6.3 Solution of the system of Wiener-Hopf equations

The system (6.5) can be solved by factorizing the symbol

$$\mathbf{H}(z,\phi,\eta) = \mathbf{I} - z\mathbf{G}(\phi,\eta), \tag{6.7}$$

i.e. for $Re(\phi) = 0$,

$$\mathbf{H}(z,\phi,\eta) = \mathbf{H}^+(z,\phi,\eta)\mathbf{H}^-(z,\phi,\eta)$$

where

$\mathbf{H}^+(z,\phi,\eta)$ is analytic for $Re(\phi) > 0$, and continuous and bounded for $Re(\phi) \geq 0$, and non-singular in $Re(\phi) > 0$.

$\mathbf{H}^-(z,\phi,\eta)$ is analytic for $Re(\phi) < 0$, and continuous and bounded for $Re(\phi) \leq 0$, and non-singular in $Re(\phi) < 0$.

To find $\mathbf{H}^+(z,\phi,\eta)$ and $\mathbf{H}^-(z,\phi,\eta)$ first we consider the following. Let

$$\mathbf{r} = \text{diag}(r_1, r_2, \cdots, r_N).$$

We then can write the matrix $\mathbf{G}(\phi,\eta)$ as

$$\mathbf{G}(\phi,\eta) = \mathbf{Ph}(\phi,\eta)(\boldsymbol{\mu}_1 + \phi\mathbf{r} + \eta\mathbf{I})^{-1}(\boldsymbol{\mu}_2 + \phi\mathbf{r} + \eta\mathbf{I})^{-1}\cdots(\boldsymbol{\mu}_m + \phi\mathbf{r} + \eta\mathbf{I})^{-1}, \tag{6.8}$$

where

$$\mathbf{h}(\phi,\eta) = \mathrm{diag}(h_1(\phi r_1 + \eta), \cdots, h_N(\phi r_N + \eta)), \tag{6.9}$$

with $h_j(\phi)$ defined on page 134, and $\boldsymbol{\mu}_i = \mathrm{diag}(\mu_{1i}, \mu_{2i}, \cdots, \mu_{Ni}), \quad i \in \mathcal{M}$.

Let

$$\alpha_{ij}(\eta) = (\eta + \mu_{ij})/r_i, \quad i \in \mathcal{N}, j \in \mathcal{M},$$

and let

$$\alpha_{ij} = \alpha_{ij}(0), \quad i \in \mathcal{N}, j \in \mathcal{M}.$$

Define $N \times N$-dimensional matrices

$$\boldsymbol{\alpha}_i(\eta) = \mathrm{diag}(\alpha_{1i}(\eta), \ldots, \alpha_{Ni}(\eta)), \quad i \in \mathcal{M},$$

$$\boldsymbol{\alpha}_i = \mathrm{diag}(\alpha_{1i}, \ldots, \alpha_{Ni}), \quad i \in \mathcal{M},$$

$$\mathbf{M}(\phi,\eta) = (\boldsymbol{\alpha}_1(\eta) + \phi\mathbf{I})(\boldsymbol{\alpha}_2(\eta) + \phi\mathbf{I})\cdots(\boldsymbol{\alpha}_m(\eta) + \phi\mathbf{I}), \tag{6.10}$$

$$\mathbf{L}(z,\phi,\eta) = \mathbf{M}(\phi,\eta) - z\mathbf{Pr}^{-m}\mathbf{h}(\phi,\eta), \tag{6.11}$$

where $\mathbf{r}^m = \underbrace{\mathbf{rr}\cdots\mathbf{r}}_{m}$ and $\mathbf{r}^{-m} = (\mathbf{r}^m)^{-1}$. It follows that

$$\mathbf{H}(z,\phi,\eta) = \mathbf{L}(z,\phi,\eta)\mathbf{M}^{-1}(\phi,\eta). \tag{6.12}$$

Furthermore,

$$\det \mathbf{L}(z,\phi,\eta) = \det \mathbf{H}(z,\phi,\eta) \det \mathbf{M}(\phi,\eta). \tag{6.13}$$

Proposition 6.3.1

1. *The poles of* $\det \mathbf{H}(z,\phi,\eta)$ *are the zeros of* $\det \mathbf{M}(\phi,\eta)$,
2. *The zeros of* $\det \mathbf{L}(z,\phi,\eta)$ *are the zeros of* $\det \mathbf{H}(z,\phi,\eta)$.

Proof. It is clear that $\det \mathbf{M}(\phi,\eta)$ has Nm zeros, i.c. $-\alpha_{11}(\eta), \cdots, -\alpha_{1m}(\eta)$, $-\alpha_{21}(\eta), \cdots,$ $-\alpha_{2m}(\eta), \cdots,$ $-\alpha_{N1}, \ldots, -\alpha_{Nm}(\eta)$. Since by definition $r_i < 0, 1 \le i \le \bar{K}$, the first $\bar{K}m$ of these lie in the right half-plane $Re(\phi) > 0$ and the $(N - \bar{K})m$ others lie in the left half-plane $Re(\phi) < 0$.

From (6.7), (6.9), (6.1), and (6.8) we see that $\det \mathbf{H}(z,\phi,\eta)$ has exactly Nm poles. Since $\det \mathbf{L}(z,\phi,\eta)$ does not have any pole, it follows from (6.13) that the Nm poles of $\det \mathbf{H}(z,\phi,\eta)$ are the zeros of $\det \mathbf{M}(\phi,\eta)$. This proves part 1 of the proposition.

It also follows from (6.13) that the zeros of $\det \mathbf{L}(z,\phi,\eta)$ are the zeros of $\det \mathbf{H}(z,\phi,\eta)$ or the zeros of $\det \mathbf{M}(\phi,\eta)$. We then can conclude that $\det \mathbf{L}(z,\phi,\eta)$ does not have any zero in common with $\det \mathbf{M}(\phi,\eta)$. This proves part 2 of the proposition.

■

Based on the proposition, we consider the following lemmas.

Lemma 6.3.1

With respect to ϕ, for $(|z| \leq 1, Re(\eta) > 0)$ or $(|z| < 1, Re(\eta) \geq 0)$, $\det \mathbf{L}(z, \phi, \eta)$ has $\bar{K}m$ zeros in the right half- plane $Re(\phi) > 0$ and has $(N - \bar{K})m$ zeros in the left half-plane $Re(\phi) < 0$.

Proof. We first study the characteristics of zeros and poles of $\det \mathbf{H}(z, \phi, \eta)$. Notice that the (i, j)th element of $z\mathbf{G}(\phi, \eta)$ is given by

$$zP_{ij} \sum_{k=1}^{m} a_{jk} \frac{\mu_{jk}}{(\phi r_j + \eta + \mu_{jk})},$$

so that $(|z| \leq 1, Re(\eta) > 0)$ or $(|z| < 1, Re(\eta) \geq 0)$,

$$\sum_{j=1}^{N} |z\mathbf{G}(\phi, \eta)_{ij}| = \sum_{j=1}^{N} |z| P_{ij} \left| \sum_{k=1}^{m} a_{jk} \frac{\mu_{jk}}{\phi r_j + \eta + \mu_{jk}} \right|. \tag{6.14}$$

We then consider the following cases, that follows from our assumption on the time the process $\{J_t\}$ spends in a state before making a transition into a different state.

Case 1 For $j \in \mathcal{N}$, H_j is hyper-exponentially distributed. On $Re(\phi) = 0$, with $(|z| \leq 1, Re(\eta) > 0)$ or $(|z| < 1, Re(\eta) \geq 0)$, it is clear that for $j \in \mathcal{N}$, $k \in \mathcal{M}$,

$$\left| \frac{\mu_{jk}}{\phi r_j + \eta + \mu_{jk}} \right| < 1. \tag{6.15}$$

Let $d_0 = (|\eta| + 2 \max_{j \in \mathcal{N}, k \in \mathcal{M}} |\mu_{jk}|) / \min_{j \in \mathcal{N}} |r_j|$. Then on $|\phi| = d$ with $d > d_0$, the inequality (6.15) is also satisfied. Since $\sum_{k=1}^{m} a_{jk} = 1$, it follows that on $C^+_{0,d}(C^-_{0,d})$,

$$\begin{aligned} \sum_{j=1}^{N} |z\mathbf{G}(\phi, \eta)_{ij}| &= \sum_{j=1}^{N} |z| P_{ij} \sum_{k=1}^{m} a_{jk} \left| \frac{\mu_{jk}}{\phi r_j + \eta + \mu_{jk}} \right| \\ &< \sum_{j=1}^{N} P_{ij} \sum_{k=1}^{m} a_{jk} \\ &= 1. \end{aligned}$$

Due to Theorem A.4.2 (generalization of Rouché's theorem) it follows that on $C^+_{0,d}(C^-_{0,d})$, the number of zeros and the number of poles inside $C^+_{0,d}$ $(C^-_{0,d})$ of $\det \mathbf{H}(z, \phi, \eta)$ are the same. From part 1 of Proposition 6.3.1 we can conclude that $\det \mathbf{H}(z, \phi, \eta)$ has $\bar{K}m$ poles in the right half- plane $Re(\phi) > 0$ and has $(N - \bar{K})m$ poles in the left half-plane $Re(\phi) < 0$. It follows that $\det \mathbf{H}(z, \phi, \eta)$ has $\bar{K}m$ zeros in the right half-plane $Re(\phi) > 0$ and has $(N - \bar{K})m$ zeros in the left half-plane $Re(\phi) < 0$. The Lemma for case 1 then can be proven by using part 2 of Proposition 6.3.1.

Case 2 For $j \in \mathcal{N}$, H_j is hypo-exponentially distributed. For this case, we use the expression for $h_j(\phi, \eta)$ given by (6.3), that is

$$h_j(\phi, \eta) = \prod_{k=1}^{m} \mu_{jk}.$$

On $Re(\phi) = 0$, with $(|z| \leq 1, Re(\eta) > 0)$ or $(|z| < 1, Re(\eta) \geq 0)$, it is clear that for $j \in \mathcal{N}$, $k \in \mathcal{M}$,

$$\left| \prod_{k=1}^{m} \frac{\mu_{jk}}{(\phi r_j + \eta + \mu_{jk})} \right| < 1. \tag{6.16}$$

Let $d_0 = (|\eta| + 2 \max_{j \in \mathcal{N}, k \in \mathcal{M}} |\mu_{jk}|) / \min_{j \in \mathcal{N}} |r_j|$. Then on $|\phi| = d$ with $d > d_0$, the inequality (6.16) is also satisfied so that on $C^+_{0,d}(C^-_{0,d})$,

$$\begin{aligned}\sum_{j=1}^{N} |z\mathbf{G}(\phi, \eta)_{ij}| &= \sum_{j=1}^{N} |z| P_{ij} \left| \prod_{k=1}^{m} \frac{\mu_{jk}}{(\phi r_j + \eta + \mu_{jk})} \right| \\ &< \sum_{j=1}^{N} P_{ij} \\ &= 1.\end{aligned}$$

Case 2 of the lemma now follows by using the same argument as in the proof of Case 1.

■

Let $\gamma_i(z, \eta)$, $i = 1, \cdots, \bar{K}m$ be the zeros of $\det \mathbf{L}(z, \phi, \eta)$ in the right half-plane $Re(\phi) > 0$ and let $\gamma_i(z, \eta)$, $i = \bar{K}m + 1, \cdots, Nm$ be the zeros in the left half-plane $Re(\phi) < 0$. The following lemma concerns the behavior of the zeros of $\det \mathbf{L}(z, \phi, 0)$ as $z \uparrow 1$.

Lemma 6.3.2
For $z \uparrow 1$ one of the $\bar{K}m$ zeros of $\det \mathbf{L}(z, \phi, 0)$ in the right half-plane $Re(\phi) > 0$ tends to 0 if and only if $\sum_{i=1}^{N} \pi_i r_i = \pi \mathbf{r} \mathbf{1} < 0$.

Proof. Let $\gamma_1(z, 0)$ be a zero of $\det \mathbf{L}(z, \phi, 0)$. Then by Proposition 6.3.1 we also have $\det \mathbf{H}(z, \gamma_1(z, 0), 0) = 0$. Let, moreover, $\mathbf{v}(z)$ be a non-zero column vector with elements $v_1(z), \cdots, v_N(z)$, satisfying $\mathbf{H}(z, \gamma_1(z, 0), 0)\mathbf{v}(z) = \mathbf{0}$. Since $\mathbf{G}(0, 0) = \mathbf{P}$, it follows that $\det \mathbf{H}(1, 0, 0) = 0$. So we may choose $\gamma_1(z, 0)$ such that $\gamma_1(1, 0) = 0$, and since

$$\mathbf{H}(1, 0, 0)\mathbf{1} = \mathbf{0}$$

we may assume that $\mathbf{v}(1) = \mathbf{1}$. Noting that $\mathbf{P}$ is irreducible it follows that $\det \mathbf{H}(1, \phi, 0)$ has a simple zero at $\phi = 0$. Consequently, the function $\gamma_1(z, 0)$ can not have a branch point at $z = 1$ and, therefore, is differentiable at $z = 1$.

From $\mathbf{H}(z, \gamma_1(z,0), 0)\mathbf{v}(z) = \mathbf{0}$ we have for $i = 1, 2, \cdots, N$,

$$\sum_{j=1}^{N}(\delta_{ij} - zG_{ij}(\gamma_1(z,0),0))v_j(z) = 0.$$

Differentiating this equation and letting z tend to 1 yields

$$\frac{dv_i(z)}{dz}\Big|_{z=1} - \sum_{j=1}^{N} P_{ij}\frac{dv_j(z)}{dz}\Big|_{z=1} - \sum_{j=1}^{N} P_{ij}v_j(1)\left[1 - r_j\sum_{k=1}^{m}\frac{a_{jk}}{\mu_{jk}}\frac{d\gamma_1(z,0)}{dz}\Big|_{z=1}\right]$$
$$= 0.$$

With our assumption $\mathbf{v}(1) = \mathbf{1}$, this can be written as

$$\frac{dv_i(z)}{dz}\Big|_{z=1} - \sum_{j=1}^{N} P_{ij}\frac{dv_j(z)}{dz}\Big|_{z=1} - \sum_{j=1}^{N} P_{ij} + \sum_{j=1}^{N} P_{ij}r_j\sum_{k=1}^{m}\frac{a_{jk}}{\mu_{jk}}\frac{d\gamma_1(z,0)}{dz}\Big|_{z=1} = 0.$$

Multiplying this equation with p_i, where $\{p_i\}$ is the stationary distribution of $\mathbf{P}$, and summing over all i we get

$$\frac{d\gamma_1(z,0)}{dz}\Big|_{z=1} = \frac{1}{\displaystyle\sum_{j=1}^{N} p_j r_j \sum_{k=1}^{m}\frac{a_{jk}}{\mu_{jk}}}.$$

Noting that

$$\pi_j = \frac{p_j\displaystyle\sum_{k=1}^{m}\frac{a_{jk}}{\mu_{jk}}}{\displaystyle\sum_{i=1}^{N} p_i\sum_{k=1}^{m}\frac{a_{ik}}{\mu_{ik}}},$$

we can now write for $z \uparrow 1$,

$$\begin{aligned}\gamma_1(z,0) &= -(1-z)\frac{d\gamma_1(z,0)}{dz}\Big|_{z=1} + o(1-z),\\ &= -(1-z)\left(\sum_{j=1}^{N}\pi_j r_j\right)\left(\sum_{j=1}^{N} p_j\sum_{k=1}^{m}\frac{a_{jk}}{\mu_{jk}}\right) + o(1-z).\end{aligned}$$

This zero lies in the right half-plane $Re(\phi) > 0$ and tends to the origin if and only if $\sum_{j=1}^{N}\pi_j r_j < 1$. ■

We next impose the following condition.

Condition 6.3.1
For $Re(\eta) \geq 0$, $-\alpha_{11}(\eta), \ldots, -\alpha_{Nm}(\eta)$ and $\gamma_1(z,\eta), \ldots, \gamma_{Nm}(z,\eta)$ are all distinct.

Similar to Condition 5.3.1, Condition 6.3.1 is needed to obtain the canonical factorization of $H(z, \phi, \eta)$. To find the factors, we first define some index sets.

$$\mathcal{A} = \{1, 2, \cdots, N\} \times \{1, 2, \cdots, m\},$$

$$\mathcal{B} = \{1, 2, \cdots, Nm\},$$

and

$$Rm^- = \{1, 2, \cdots, \bar{K}m\}.$$

We define a map t_m from $\mathcal{B}$ to $\mathcal{A}$

$$t_m : l \longmapsto \left(\left\lfloor \frac{l-1}{m} \right\rfloor + 1, l - m \left\lfloor \frac{l-1}{m} \right\rfloor \right),$$

where $\lfloor a \rfloor$ denotes the largest integer less or equal to a. This map defines a one to one correspondence from $\mathcal{B}$ to $\mathcal{A}$, with inverse $t_m^{-1} : (i, j) \to m(i-1) + j$. Notice that this inverse gives an enumeration of the set $\mathcal{A}$. We also define the map

$$t_{\bar{K}} : l \longmapsto \left\lfloor \frac{l-1}{\bar{K}} \right\rfloor + 1,$$

which defines a function from Rm^- to $\{1, 2, \cdots, m\}$. We will use this function to divide Rm^- into m sets with cardinality $\bar{K}$.

For $i \in \mathcal{B}$, let $\mathbf{E}^i(z, \eta)$ be a non-unique nonzero column vector satisfying

$$\mathbf{L}(z, \gamma_i(z, \eta), \eta)\mathbf{E}^i(z, \eta) = 0, \tag{6.17}$$

and let $\mathbf{E}(z, \eta)$ be the $N \times Nm$-matrix with ith column is $\mathbf{E}^i(z, \eta)$.

For $|z| \leq 1, Re(\eta) \geq 0$, let $\mathbf{D}(z, \eta)$ be the $N \times \bar{K}m$-matrix with elements

$$\begin{aligned} &D_{ij}(z, \eta) \\ =&(\alpha_{i1}(\eta) + \gamma_j(z, \eta))(\alpha_{i2}(\eta) + \gamma_j(z, \eta)) \cdots (\alpha_{im}(\eta) + \gamma_j(z, \eta))E_{ij}(z, \eta), \end{aligned}$$

$i \in \mathcal{N}; j \in Rm^-$. The ith column of matrix $\mathbf{D}(z, \eta)$ satisfies

$$\mathbf{D}^i(z, \eta) = \mathbf{M}(\gamma_i(z, \eta), \eta)\mathbf{E}^i(z, \eta). \tag{6.18}$$

It follows that for $i \in Rm^-$,

$$\begin{aligned} \mathbf{H}(z, \gamma_i(z, \eta), \eta)\mathbf{D}^i(z, \eta) &= \mathbf{L}(z, \gamma_i(z, \eta), \eta)\mathbf{M}^{-1}(\gamma_i(z, \eta), \eta)\mathbf{M}(\gamma_i(z, \eta), \eta)\mathbf{E}^i(z, \eta) \\ &= \mathbf{0}. \end{aligned} \tag{6.19}$$

For $i \in \mathcal{N}$, $|z| \leq 1, Re(\eta) \geq 0$, define the $\bar{K} \times \bar{K}m$-matrix ${}^i\mathbf{S}(z, \eta)$ with elements

$${}^iS_{jk}(z, \eta) = \prod_{l=1, l \neq i}^{m} (\alpha_{jl}(\eta) + \gamma_k(z, \eta))E_{jk}(z, \eta), \quad j \in R^-, k \in Rm^-. \tag{6.20}$$

Let $\mathbf{S}(z,\eta)$ be the $\bar{K}m \times \bar{K}m$-dimensional matrix so that its j-th row is the $j - \lfloor \frac{j-1}{\bar{K}} \rfloor$-th row of matrix ${}^{t_{\bar{K}}(j)}S(z,\eta)$, or

$$\mathbf{S}(z,\eta) = \begin{pmatrix} {}^1S(z,\eta) \\ {}^2S(z,\eta) \\ \vdots \\ {}^mS(z,\eta) \end{pmatrix}.$$

We impose the following condition.

Condition 6.3.2
$\det \mathbf{S}(z,\eta) \neq 0$ *for* $z \neq 0$ *and* $Re(\eta) \geq 0$.

Let $\mathbf{C}_0$ be the $\bar{K}m \times N$-dimensional matrix defined by

$$\mathbf{C}_0 = \begin{pmatrix} \mathbf{I}_{\bar{K}N} \\ \mathbf{I}_{\bar{K}N} \\ \vdots \\ \mathbf{I}_{\bar{K}N} \end{pmatrix},$$

where $\mathbf{I}_{\bar{K}N}$ is the $\bar{K} \times N$-dimensional matrix defined on page 136. Define the $\bar{K}m \times N$-matrix $\mathbf{C}(z,\eta)$ by

$$\mathbf{C}(z,\eta) = \mathbf{S}^{-1}(z,\eta)\mathbf{C}_0, \quad |z| \leq 1, Re(\eta) \geq 0. \tag{6.21}$$

Notice that the last $N - \bar{K}$ columns of $\mathbf{C}(z,\eta)$ are equal zero. Now, define for $(|z| < 1, Re(\eta) \geq 0, Re(\phi) \geq 0)$ or $(|z| \leq 1, Re(\eta) > 0, Re(\phi) \geq 0)$ or $(|z| \leq 1, Re(\eta) \geq 0, Re(\phi) > 0)$, the $N \times N$-matrix $\mathbf{K}(z,\phi,\eta)$ by

$$\mathbf{K}(\mathbf{z},\phi,\eta) = \mathbf{I} + \mathbf{D}(z,\eta)\mathrm{diag}\left(\frac{1}{\phi - \gamma_1(z,\eta)}, \ldots, \frac{1}{\phi - \gamma_{\bar{K}m}(z,\eta)}\right)\mathbf{C}(z,\eta). \tag{6.22}$$

We now give the explicit factorization theorem.

Theorem 6.3.1
If Condition 6.3.1 and 6.3.2 are satisfied then for $(|z| < 1, Re(\eta) \geq 0)$ *or* $(|z| \leq 1, Re(\eta) > 0)$,

1. $\det \mathbf{K}(\mathbf{z},\phi,\eta) = \prod_{i=1}^{\bar{K}m} \left(\frac{\phi + \alpha_{t_m(i)}(\eta)}{\phi - \gamma_i(z,\eta)}\right)$,
2. *for* $Re(\phi) = 0$
   $\mathbf{H}(z,\phi,\eta) = \mathbf{H}^+(z,\phi,\eta)\mathbf{H}^-(z,\phi,\eta)$
   where
   (a) $\mathbf{H}^-(z,\phi,\eta) = \mathbf{K}^{-1}(z,\phi,\eta)$
   (b) $\mathbf{H}^+(z,\phi,\eta) = \mathbf{H}(z,\phi,\eta)\mathbf{K}(z,\phi,\eta)$

(c) $\mathbf{H}^{+}(z,\phi,\eta)$ *is analytic for* $Re(\phi)>0$*, and continuous and bounded for* $Re(\phi)\geq 0$*, and non-singular in* $Re(\phi)>0$*,*
$\mathbf{H}^{-}(z,\phi,\eta)$ *is analytic for* $Re(\phi)<0$*, and continuous and bounded for* $Re(\phi)\leq 0$*, and non-singular in* $Re(\phi)<0$*.*

Proof.
By rearranging the diagonal elements of matrix $\mathbf{M}^{-1}(\phi,\eta)$ into its partial fractions we have that

$$\begin{aligned}\mathbf{H}^{+}(z,\phi,\eta)&=\mathbf{H}(z,\phi,\eta)\mathbf{K}(z,\phi,\eta)\\&=\mathbf{K}(\mathbf{z},\phi,\eta)-z\mathbf{Ph}(\phi,\eta)\mathbf{r}^{-m}\sum_{i=1}^{N}\frac{\mathbf{1}_i^T\mathbf{1}_i}{\prod_{j=1}^{m}(\phi+\alpha_{ij}(\eta))}\mathbf{K}(\mathbf{z},\phi,\eta)\\&=\mathbf{K}(\mathbf{z},\phi,\eta)-z\mathbf{Ph}(\phi,\eta)r^{-m}\sum_{i=1}^{N}\sum_{j=1}^{m}\frac{A_{ij}(\eta)\mathcal{I}_{ij}\mathcal{I}_{ij}^T}{(\phi+\alpha_{ij}(\eta))}\mathbf{K}(\mathbf{z},\phi,\eta),\end{aligned}\tag{6.23}$$

where

- $A_{ij}(\eta), i=1,2,\cdots,N, j=1,2,\cdots,m$ are constants not depending on ϕ,
- $\mathbf{1}_i$ is the $N-$dimensional row vector with ith component 1 and all other components equal to 0,
- $\mathcal{I}_{ij}$ is the $N\times Nm$-matrix with $(i,(j-1)N+i)$ -th element 1 and all other components equal to 0.

Moreover, for $l\in Rm^{-}$ with $t_m(l)=(i_1,i_2)$ we have using (6.22), (6.18), (6.21), and (6.20),

$$\begin{aligned}&\mathcal{I}_{i_1i_2}^T\mathbf{K}(z,-\alpha_{i_1i_2}(\eta),\eta)\\&=\mathcal{I}_{i_1i_2}^T-\mathcal{I}_{i_1i_2}^T\sum_{j=1}^{\bar{K}m}\mathbf{D}^j(z,\eta)\mathbf{C}_j(z,\eta)/(\alpha_{i_1i_2}(\eta)+\gamma_j(z,\eta))\\&=\mathcal{I}_{i_1i_2}^T-\bar{\mathbf{1}}_{(i_2-1)N+i_1}^T\sum_{j=1}^{\bar{K}m}\frac{\prod_{k=1}^{m}(\alpha_{i_1k}(\eta)+\gamma_j(z,\eta))}{(\alpha_{i_1i_2}(\eta)+\gamma_j(z,\eta))}E_{i_1j}(z,\eta)\mathbf{C}_j(z,\eta)\\&=\mathcal{I}_{i_1i_2}^T-\bar{\mathbf{1}}_{(i_2-1)N+i_1}^T\sum_{j=1}^{\bar{K}m}{}^{i_2}S_{i_1j}(z,\eta)\mathbf{S}_j^{-1}(z,\eta)C_0I_{\bar{K}N}\\&=\mathcal{I}_{i_1i_2}^T-\bar{\mathbf{1}}_{(i_2-1)N+i_1}^T\mathbf{1}_{i_1}\\&=\mathcal{I}_{i_1i_2}^T-\mathcal{I}_{i_1i_2}^T\\&=\mathbf{0},\end{aligned}$$

where $\bar{\mathbf{1}}_i$ is the $\bar{K}m$-dimensional row vector with ith component 1 and all other components equal to 0. This yields for $l\in Rm^{-}$,

$$\lim_{\phi\to-\alpha_{t_m(l)}}(\phi+\alpha_{t_m(l)})\mathbf{H}^{+}(z,\phi,\eta)=0,$$

and this shows that $\mathbf{H}^{+}(z,\phi,\eta)$ has no pole at $\phi = -\alpha_{t_m(l)}, l \in Rm^{-}$.

As in the proof of Theorem 5.3.1 in chapter 5 we can show that

$$\det \mathbf{K}(\mathbf{z},\phi,\eta) = \prod_{i=1}^{\bar{K}m} \left(\frac{\phi + \alpha_{t_m(i)}(\eta)}{\phi - \gamma_i(z,\eta)} \right),$$

which proves part 1 of the theorem.

We see from (6.22) that

$$\mathbf{H}^{+}(z,\phi,\eta) = \mathbf{H}(z,\phi,\eta) + \mathbf{H}(z,\phi,\eta) \sum_{j=1}^{\bar{K}m} \frac{\mathbf{D}^{j}(z,\eta)\mathbf{C}_j(z,\eta)}{(\phi - \gamma_j(z,\eta))},$$

so from (6.7), Condition 6.3.1, and (6.19) it follows that for $i \in Rm^{-}$,

$$\lim_{\phi \to \gamma_i(z,\eta)} (\phi - \gamma_i(z,\eta))\mathbf{H}^{+}(z,\phi,\eta) = \mathbf{H}(z,\gamma_i(z,\eta),\eta)\mathbf{D}^{i}(z,\eta)\mathbf{C}_i(z,\eta) = \mathbf{0},$$

which proves part 2 of the theorem. ■

With this result we can write for the system (6.5)

$$\mathbf{Z}(z,\phi,\eta,v)\mathbf{H}(z,\phi,\eta)\mathbf{K}(z,\phi,\eta) = z\mathbf{Z}^{0}(\phi,\eta,v)\mathbf{K}(z,\phi,\eta) + \mathbf{V}(z,\phi,\eta,v)\mathbf{K}(z,\phi,\eta), \quad (6.24)$$

where the left-hand side is analytic in $Re(\phi) > 0$ and bounded and continuous in $Re(\phi) \geq 0$, and the last term of the right-hand side is analytic in $Re(\phi) < 0$ and bounded and continuous in $Re(\phi) \leq 0$.

Next we decompose the first term of the right-hand side, we determine functions $\mathbf{K}^{+}$ and $\mathbf{K}^{-}$ such that i.e. for $Re(\phi) = 0$,

$$\mathbf{Z}^{0}(\phi,\eta,v)\mathbf{K}(z,\phi,\eta) = \mathbf{K}^{+}(z,\phi,\eta,v) + \mathbf{K}^{-}(z,\phi,\eta,v) \quad (6.25)$$

where

$\mathbf{K}^{+}(z,\phi,\eta,v)$ is analytic for $Re(\phi) > 0$, and continuous and bounded for $Re(\phi) \geq 0$,

$\mathbf{K}^{-}(z,\phi,\eta,v)$ is analytic for $Re(\phi) < 0$, and continuous and bounded for $Re(\phi) \leq 0$.

Lemma 6.3.3
If Conditions 6.3.1 and 6.3.2 are satisfied then for $Re(\phi) \geq 0$,

$$\begin{aligned} & K_{ij}^{+}(z,\phi,\eta,v) \\ = \; & \delta_{ij} Z_i^{0}(\phi,\eta,v) + \sum_{k=1}^{\bar{K}m} D_{ik}(z,\eta) \frac{Z_i^{0}(\phi,\eta,v) - Z_i^{0}(\gamma_k(z,\eta),\eta,v)}{\phi - \gamma_k(z,\eta)} C_{kj}(z,\eta) \end{aligned}$$

(6.26)

and for $Re(\phi) \leq 0$,

$$K_{ij}^{-}(z,\phi,\eta,v) = \sum_{k=1}^{\bar{K}m} D_{ik}(z,\eta)\frac{Z_i^0(\gamma_k(z,\eta),\eta,v)}{\phi - \gamma_k(z,\eta)}C_{kj}(z,\eta) \tag{6.27}$$

if $(|z| < 1, Re(\eta) \geq 0)$ *or* $(|z| \leq 1, Re(\eta) > 0)$.

Proof. Since by definition the zeroes $\gamma_k(z,\eta), i = 1,2,\cdots,\bar{K}m$ all lie in the right half-plane $Re(\phi) \geq 0$, it is clear that for $i,j \in \mathcal{N}$, $K_{ij}^{-}(z,\phi,\eta,v)$ does not have any pole in the left half-plane $Re(\phi) < 0$.

For $l = 1,2,\cdots,\bar{K}m$ it follows from (6.18), (6.20) and Condition 6.3.1 that

$$\begin{aligned}
&\lim_{\phi\to\gamma_l(z,\eta)} (\phi - \gamma_l(z,\eta))K_{ij}^{+}(z,\phi,\eta,v) \\
&= \lim_{\phi\to\gamma_l(z,\eta)} (\phi - \gamma_l(z,\eta))\delta_{ij}Z_i^0(\phi,\eta,v) \\
&+ \lim_{\phi\to\gamma_l(z,\eta)} (\phi - \gamma_l(z,\eta))\sum_{k=1}^{\bar{K}m} D_{ik}(z,\eta)\frac{Z_i^0(\phi,\eta,v) - Z_i^0(\gamma_k(z,\eta),\eta,v)}{\phi - \gamma_k(z,\eta)}C_{kj}(z,\eta) \\
&= \lim_{\phi\to\gamma_l(z,\eta)} (\phi - \gamma_l(z,\eta))\delta_{ij}Z_i^0(\phi,\eta,v),
\end{aligned}$$

where the expression for $Z_i^0(\phi,\eta,v)$ is given by (6.6). It follows that for $l = 1,2,\cdots,\bar{K}m$, $i,j \in \mathcal{N}$,

$$\lim_{\phi\to\gamma_l(z,\eta)} (\phi - \gamma_l(z,\eta))K_{ij}^{+}(z,\phi,\eta,v) = 0,$$

thus $K_{ij}^{+}(z,\phi,\eta,v)$ is analytic in the right half-plane $Re(\phi) > 0$.

■

Theorem 6.3.2
If conditions 6.3.1 and 6.3.2 are satisfied then for $Re(\phi) \geq 0$,

$$\mathbf{Z}(z,\phi,\eta,v)\mathbf{H}(z,\phi,\eta)\mathbf{K}(z,\phi,\eta) = z\mathbf{K}^{+}(z,\phi,\eta,v) + z\mathbf{K}^{-}(z,0,\eta,v) \tag{6.28}$$

if $(|z| < 1, Re(\eta) \geq 0)$ *or* $(|z| \leq 1, Re(\eta) > 0)$.

Proof. See the proof of Theorem 5.3.2 in chapter 5.

■

In section 6.4 we will study the distribution of the buffer content. We will see that the expression for the distribution functions of interest can be obtained once we find an explicit expression for $\mathbf{Z}(1,\phi,\eta,v)$, which can be easily found by multiplying both sides of (6.28) with $\mathbf{H}^{+}(z,\phi,\eta)^{-1} = [\mathbf{H}(z,\phi,\eta)\mathbf{K}(z,\phi,\eta)]^{-1}$. Lemma 6.3.5 below gives us an expression for $\mathbf{H}^{+}(z,\phi,\eta)^{-1}$, and in the following we first define some vectors and matrices that will be used in the lemma.

By definition,

$$\mathbf{H}(z,\phi,\eta)^{-1} = \mathbf{M}(\phi,\eta)\mathbf{L}(1,\phi,\eta)^{-1}.$$

To obtain an explicit expression for $\mathbf{L}(1,\phi,\eta)^{-1}$, first we rewrite the matrix $\mathbf{L}(1,\phi,\eta)$ as

$$\mathbf{L}(1,\phi,\eta) = \phi^m \mathbf{I} + \sum_{i=0}^{m-1} \phi^i \mathbf{L}_i(\eta).$$

Due to theorem A.5.1 on page 189 in the appendix, the expression for $\mathbf{L}(1,\phi,\eta)^{-1}$ is given by

$$\mathbf{L}(1,\phi,\eta)^{-1} = \mathbf{P}_1(\phi\mathbf{I} - \mathbf{C}_L(\eta))^{-1}\mathbf{R}_1, \tag{6.29}$$

where

$$\mathbf{P}_1 = (\mathbf{I}\,\mathbf{0}\cdots\mathbf{0}), \quad \mathbf{R}_1 = \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{I} \end{pmatrix},$$

and the (first) companion matrix of $\mathbf{L}(1,\phi,\eta)$ is defined by

$$\mathbf{C}_L = \begin{pmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \mathbf{I} \\ -\mathbf{L}_0(\eta) & -\mathbf{L}_1(\eta) & -\mathbf{L}_2(\eta) & \cdots & -\mathbf{L}_{m-1}(\eta) \end{pmatrix}.$$

Let

$$\boldsymbol{\gamma}(\eta) = \mathrm{diag}(\gamma_1(1,\eta), \cdots, \gamma_{Nm}(1,\eta)),$$

and let

$$\widetilde{\mathbf{E}}(\eta) = (\mathbf{e}^1(\eta)\cdots\mathbf{e}^{Nm}(\eta)),$$

where $\mathbf{e}^i(\eta)$ is the eigenvector that corresponds to the eigenvalue $\gamma_i(1,\eta)$ of $\mathbf{C}_L$ (note that the eigenvalues of $\mathbf{C}_L$ are also the zeros of $\det\mathbf{L}(1,\phi,\eta)$). If condition 6.3.1 is satisfied, the rank of $\widetilde{\mathbf{E}}(\eta)$ is Nm, and we can write

$$\boldsymbol{\gamma}(\eta) = \widetilde{\mathbf{E}}(\eta)^{-1}\mathbf{C}_L(\eta)\widetilde{\mathbf{E}}(\eta).$$

It follows that

$$\begin{aligned} \mathbf{L}(1,\phi,\eta)^{-1} =& \mathbf{E}(\eta)(\phi\mathbf{I} - \boldsymbol{\gamma}(\eta))^{-1}\mathbf{Y}(\eta) \\ =& \sum_{i=1}^{Nm} \frac{\mathbf{E}^i(\eta), \mathbf{Y}_i(\eta)}{(\phi - \gamma_i(1,\eta))}, \end{aligned} \tag{6.30}$$

where

$$\mathbf{E}(\eta) = \mathbf{P}_1\widetilde{\mathbf{E}}(\eta), \quad \mathbf{Y}(\eta) = \widetilde{\mathbf{E}}(\eta)^{-1}\mathbf{R}_1,$$

with $\mathbf{E}^i(\eta)(\mathbf{Y}_i(\eta))$ denoting the ith column(row) of matrix $\mathbf{E}(\eta)(\mathbf{Y}(\eta))$. It can be easily shown that the matrix $\mathbf{E}^i(\eta)$ is the same as $\mathbf{E}(1,\eta)$ defined on page 143. For brevity, in the rest of this chapter we will use the notation $\mathbf{E}^i(\eta)$ instead of $\mathbf{E}(1,\eta)$.

To obtain an explicit expression for $\mathbf{K}(z,\phi,\eta)^{-1}$, define for $Re(\phi)\geq 0, Re(\eta)\geq 0$, the $\bar{K}m\times\bar{K}m$-matrix $\mathbf{X}(\phi,\eta)$ by

$$\mathbf{X}(\phi,\eta)=\phi\tilde{\mathbf{I}}-\mathrm{diag}(\gamma_1(1,\eta),\ldots,\gamma_{\bar{K}m}(1,\eta))+\mathbf{C}(1,\eta)\mathbf{D}(1,\eta), \tag{6.31}$$

where $\tilde{\mathbf{I}}$ is the $\bar{K}m\times\bar{K}m$ identity matrix.

Lemma 6.3.4
For $Re(\phi)\geq 0, Re(\eta)\geq 0$,

$$\mathbf{X}(\phi,\eta)^{-1}=\mathbf{S}(1,\eta)^{-1}(\phi\tilde{\mathbf{I}}+\widetilde{\boldsymbol{\alpha}}(\eta))^{-1}\mathbf{S}(1,\eta), \tag{6.32}$$

where

$$\widetilde{\boldsymbol{\alpha}}(\eta)=diag(\alpha_{11}(\eta),\alpha_{21}(\eta),\cdots,\alpha_{\bar{K}1}(\eta),\alpha_{12}(\eta),\cdots,\alpha_{\bar{K}2}(\eta),\cdots,\alpha_{\bar{K}m}(\eta)). \tag{6.33}$$

Proof. See page 194 in the appendix.

Now we have the explicit expression for $\mathbf{H}^+(1,\phi,\eta)^{-1}$ which is given in the following lemma.

Lemma 6.3.5
If Conditions 6.3.1 and 6.3.2 are satisfied then for $Re(\phi)\geq 0$, $Re(\eta)>0$,

$$\begin{aligned}&\mathbf{H}^+(1,\phi,\eta)^{-1}\\&=\Big(\mathbf{I}-\mathbf{D}(1,\eta)\mathbf{S}(1,\eta)^{-1}(\phi\tilde{\mathbf{I}}+\widetilde{\boldsymbol{\alpha}}(\eta))^{-1}\mathbf{C}_0\Big)\mathbf{M}(\phi,\eta)\sum_{k=1}^{Nm}\frac{\mathbf{E}^k(\eta)Y_k(\eta)}{(\phi-\gamma_k(1,\eta))}.\end{aligned} \tag{6.34}$$

Proof. If Conditions 6.3.1 and 6.3.2 are satisfied then from (6.12) and (6.30) we obtain for $Re(\phi)\geq 0$, $Re(\eta)>0$,

$$\mathbf{H}(1,\phi,\eta)^{-1}=\mathbf{M}(\phi,\eta)\sum_{i=1}^{Nm}\frac{\mathbf{E}^i(\eta)Y_i(\eta)}{(\phi-\gamma_i(1,\eta))}. \tag{6.35}$$

Moreover, from (6.22), (A.10) in the appendix, and (6.32) we also have for $Re(\phi)\geq 0$, $Re(\eta)>0$,

$$\begin{aligned}&\mathbf{K}(1,\phi,\eta)^{-1}\\=&\mathbf{I}-\mathbf{D}(1,\eta)\mathbf{X}(\phi,\eta)^{-1}\mathbf{C}(1,\eta)\\=&\mathbf{I}-\mathbf{D}(1,\eta)\mathbf{S}(1,\eta)^{-1}(\phi\tilde{\mathbf{I}}+\widetilde{\boldsymbol{\alpha}}(\eta))^{-1}\mathbf{S}(1,\eta)\mathbf{C}(1,\eta)\\=&\mathbf{I}-\mathbf{D}(1,\eta)\mathbf{S}(1,\eta)^{-1}(\phi\mathbf{I}+\widetilde{\boldsymbol{\alpha}}(\eta))^{-1}\mathbf{C}_0.\end{aligned} \tag{6.36}$$

Since $\mathbf{H}^+(1,\phi,\eta)^{-1}=\mathbf{K}(1,\phi,\eta)^{-1}\mathbf{H}(1,\phi,\eta)^{-1}$, then from (6.36) and (6.35) we obtain (6.34), and it proves the lemma.

∎

In order to have closed-form expressions for the Laplace-Stieltjes transforms of probability distributions of interest, in the following we will rewrite the rational matrix $Z(1, \phi, \eta, v)$ in a form so that in each element the degree of the numerator is less than the degree of the denominator. In doing so, we note that for every positive integer n,

$$\prod_{k=1}^{n}(\phi + a_k) = \sum_{k=0}^{n} \phi^k c_k, \tag{6.37}$$

where the coefficients c_{n-k} are given by

$$c_n = 1, \tag{6.38}$$

$$c_{n-k} = \sum_{l_1, l_2, \cdots, l_k \in \mathcal{C}_k^n} \prod_{j=1}^{k} a_{l_j}, \tag{6.39}$$

where $\mathcal{C}_k^n$ is the set of combinations of k elements out of $\{1, 2, \cdots, n\}$, $k = 1, 2, \cdots, n$.

Besides the rearrangement (6.37), we also should consider the following lemma.

Lemma 6.3.6

For $l = 1, \cdots, m-1$,

$$\sum_{k=1}^{Nm} \frac{\phi^l \mathbf{E}^k(\eta) Y_k(\eta)}{(\phi - \gamma_k(1, \eta))} = \sum_{k=1}^{Nm} \frac{\gamma_k(1, \eta)^l \mathbf{E}^k(\eta) Y_k(\eta)}{(\phi - \gamma_k(1, \eta))},$$

and

$$\sum_{k=1}^{Nm} \frac{\phi^m \mathbf{E}^k(\eta) Y_k(\eta)}{(\phi - \gamma_k(1, \eta))} = \mathbf{I} + \sum_{k=1}^{Nm} \frac{\gamma_k(1, \eta)^m \mathbf{E}^k(\eta) Y_k(\eta)}{(\phi - \gamma_k(1, \eta))}.$$

Proof. The matrices $\mathbf{E}(\eta)$ and $\mathbf{Y}(\eta)$ have the property (see page 52 of Gohberg *et al.*[28] for the proof) that

$$\mathbf{E}(\eta)\boldsymbol{\gamma}^k \mathbf{Y}(\eta) = \begin{cases} \mathbf{0} & \text{, for } k = 0, \cdots, m-2, \\ \mathbf{I} & \text{, for } k = m-1, \end{cases}$$

so that

$$\begin{aligned} \sum_{k=1}^{Nm} \frac{\phi \mathbf{E}^k(\eta) Y_k(\eta)}{(\phi - \gamma_k(1, \eta))} &= \sum_{k=1}^{Nm} \frac{(\phi - \gamma_k(1, \eta) + \gamma_k(1, \eta)) \mathbf{E}^k(\eta) Y_k(\eta)}{(\phi - \gamma_k(1, \eta))} \\ &= \sum_{k=1}^{Nm} \mathbf{E}^k(\eta) Y_k(\eta) + \sum_{k=1}^{Nm} \frac{\gamma_k(1, \eta) \mathbf{E}^k(\eta) Y_k(\eta)}{(\phi - \gamma_k(1, \eta))} \\ &= \sum_{k=1}^{Nm} \frac{\gamma_k(1, \eta) \mathbf{E}^k(\eta) Y_k(\eta)}{(\phi - \gamma_k(1, \eta))}. \end{aligned}$$

Furthermore, we can prove by induction that for $l = 1, \cdots, m-1$,

$$\sum_{k=1}^{Nm} \frac{\phi^l \mathbf{E}^k(\eta) Y_k(\eta)}{(\phi - \gamma_k(1,\eta))} = \sum_{k=1}^{Nm} \frac{\gamma_k(1,\eta)^l \mathbf{E}^k(\eta) Y_k(\eta)}{(\phi - \gamma_k(1,\eta))}.$$

For $l = m$, it follows that

$$\begin{aligned}\sum_{k=1}^{Nm} \frac{\phi^m \mathbf{E}^k(\eta) Y_k(\eta)}{(\phi - \gamma_k(1,\eta))} &= \phi \sum_{k=1}^{Nm} \frac{\phi^{m-1} \mathbf{E}^k(\eta) Y_k(\eta)}{(\phi - \gamma_k(1,\eta))} \\ &= \phi \sum_{k=1}^{Nm} \frac{\gamma_k(1,\eta)^{m-1} \mathbf{E}^k(\eta) Y_k(\eta)}{(\phi - \gamma_k(1,\eta))} \\ &= \mathbf{I} + \sum_{k=1}^{Nm} \frac{\gamma_k(1,\eta)^{m} \mathbf{E}^k(\eta) Y_k(\eta)}{(\phi - \gamma_k(1,\eta))}.\end{aligned}$$

■

With the explicit expression for $\mathbf{H}^+(1,\phi,\eta)^{-1}$ given by (6.34), together with (6.37) and Lemma 6.3.6, we are halfway to an explicit expression for $Z(1,\phi,\eta,v)$.

From (6.28) we obtain $Re(\phi) \geq 0, Re(\eta) > 0$,

$$\mathbf{Z}(1,\phi,\eta,v) = \left(\mathbf{K}^+(1,\phi,\eta,v) + \mathbf{K}^-(1,0,\eta,v)\right) \mathbf{H}^+(1,\phi,\eta)^{-1}.$$

Now, using (6.25) and by definition of $K^-_{ij}(1,\phi,\eta,v)$, $\mathbf{K}^+(1,\phi,\eta,v) + \mathbf{K}^-(1,0,\eta,v)$ can be rewritten as

$$\mathbf{Z}^0(\phi,\eta,v)\mathbf{K}(\phi,\eta,v) - \phi \sum_{i=1}^{\bar{K}m} \frac{\mathbf{Z}^0(\gamma_i(1,\eta),\eta,v)\mathbf{D}^i(1,\eta)\mathbf{C}_i(1,\eta)}{\gamma_i(1,\eta)(\phi - \gamma_i(1,\eta))},$$

so that by multiplying with $\mathbf{H}^+(1,\phi,\eta)^{-1}$, as given by (6.34), yields for $Re(\phi) \geq 0$, $Re(\eta) > 0$,

$$\begin{aligned}&\mathbf{Z}(1,\phi,\eta,v) \\ =&\mathbf{Z}^0(\phi,\eta,v)\mathbf{M}(\phi,\eta) \sum_{i=1}^{Nm} \frac{\mathbf{E}^i(\eta) Y_i(\eta)}{(\phi - \gamma_i(1,\eta))} \\ &- \phi \sum_{i=1}^{\bar{K}m} \frac{\mathbf{Z}^0(\gamma_i(1,\eta),\eta,v)\mathbf{D}^i(1,\eta)\mathbf{C}_i(1,\eta)}{\gamma_i(1,\eta)(\phi - \gamma_i(1,\eta))} \left(\mathbf{I} - \mathbf{D}(1,\eta)\mathbf{X}(\phi,\eta)^{-1}\mathbf{C}(1,\eta)\right) \\ &.\mathbf{M}(\phi,\eta) \sum_{k=1}^{Nm} \frac{\mathbf{E}^k(\eta) Y_k(\eta)}{(\phi - \gamma_k(1,\eta))}.\end{aligned} \tag{6.40}$$

Using (6.31) we see that

$$\begin{aligned}&\mathbf{C}_i(1,\eta)\left(\mathbf{I} - \mathbf{D}(1,\eta)\mathbf{X}^{-1}(\phi,\eta)\mathbf{C}(1,\eta)\right) \\ =&\mathbf{C}_i(1,\eta) - \left(\mathbf{C}_i(1,\eta)\mathbf{D}(1,\eta) + (\phi - \gamma_i(1,\eta))\mathbf{1}_i\right)\mathbf{X}^{-1}(\phi,\eta)\mathbf{C}(1,\eta) \\ &\quad + (\phi - \gamma_i(1,\eta))\mathbf{1}_i\mathbf{X}^{-1}(\phi,\eta)\mathbf{C}(1,\eta) \\ =&\mathbf{C}_i(1,\eta) - \mathbf{C}_i(1,\eta) - (\phi - \gamma_i(1,\eta))\mathbf{1}_i\mathbf{X}^{-1}(\phi,\eta)\mathbf{C}(1,\eta) \\ =&(\phi - \gamma_i(1,\eta))\mathbf{1}_i\mathbf{X}^{-1}(\phi,\eta)\mathbf{C}(1,\eta),\end{aligned} \tag{6.41}$$

and by substituting this to (6.40) and by using (6.32) we obtain for $Re(\phi) \geq 0$, $Re(\eta) > 0$,

$$\begin{aligned}
&\mathbf{Z}(1,\phi,\eta,v)\\
=&\,\mathbf{Z}^0(\phi,\eta,v)\mathbf{M}(\phi,\eta)\sum_{k=1}^{Nm}\frac{\mathbf{E}^k(\eta)Y_k(\eta)}{(\phi-\gamma_k(1,\eta))}\\
&-\phi\sum_{i=1}^{\bar{K}m}\frac{\mathbf{Z}^0(\gamma_i(1,\eta),\eta,v)\mathbf{D}^i(1,\eta)(\mathbf{S}(1,\eta)^{-1})_i}{\gamma_i(1,\eta)}\left[\phi\tilde{\mathbf{I}}+\tilde{\boldsymbol{\alpha}}(\eta)\right]^{-1}\mathbf{C}_0\mathbf{M}(\phi,\eta)\\
&\cdot\sum_{k=1}^{Nm}\frac{\mathbf{E}^k(\eta)Y_k(\eta)}{(\phi-\gamma_k(1,\eta))}.
\end{aligned} \tag{6.42}$$

In Appendix A.10 we check that $\mathbf{Z}(1,\phi,\eta,v)$ is indeed analytic in the right half-plane $Re(\phi) \geq 0$.

In the following we will use the rearrangement (6.37) and Lemma 6.3.7 to rewrite the right hand-side of (6.42) so that in each term, the degree of the numerator is less than the degree of the denominator.

For $j = 1, \cdots, m$, define the map

$$\theta_j : \mathcal{M} \setminus \{j\} \mapsto \{1, 2, \cdots, m-1\},$$

where

$$\theta_j(k) = \begin{cases} k & , k \leq j-1, \\ k-1 & , k > j. \end{cases}$$

For brevity, we will often write by $\theta_j = k$ for $\theta_j(l) = k$, $l \in \mathcal{M} \setminus \{j\}$.

We first consider the matrix

$$\mathbf{Z}^0(\phi,\eta,v)\mathbf{M}(\phi,\eta)\sum_{k=1}^{Nm}\frac{\mathbf{E}^k(\eta)Y_k(\eta)}{(\phi-\gamma_k(1,\eta))}.$$

From (6.6) and (6.10) we obtain for $Re(\phi) \geq 0$, $Re(\eta) \geq 0$,

$$\mathbf{Z}^0(\phi,\eta,v)\mathbf{M}(\phi,\eta) = \mathbf{A}_0(\eta)\mathbf{M}(\phi,\eta) + \sum_{j=1}^{m}\mathbf{A}_j(\phi,\eta)\widetilde{\mathbf{M}}_j(\phi,\eta), \tag{6.43}$$

where

$$\mathbf{A}_0(\eta) = \mathrm{diag}\left(\sum_{k=1}^{m}\frac{a_{1k}\mu_{1k}e^{\alpha_{1k}(\eta)v}}{r_1\alpha_{1k}(\eta)}, \cdots, \sum_{k=1}^{m}\frac{a_{\bar{K}k}\mu_{\bar{K}k}e^{\alpha_{\bar{K}k}(\eta)v}}{r_{\bar{K}}\alpha_{\bar{K}k}(\eta)}, 0, \cdots, 0\right),$$

and for $j = 1, \cdots, m$,

$$\mathbf{A}_j(\phi,\eta) = \mathrm{diag}\left(\frac{a_{1j}\mu_{1j}e_{1j}(\phi,\eta)}{r_1}, \cdots, \frac{a_{Nj}\mu_{Nj}e_{1N}(\phi,\eta)}{r_N}\right),$$

with

$$e_{ij}(\phi,\eta) = \begin{cases} e^{-\phi v} - e^{\alpha_{ij}(\eta)v}, & \text{for } j \le \bar{K}, \\ e^{-\phi v}, & \text{for } j > \bar{K}. \end{cases}$$

and

$$\widetilde{\mathbf{M}}_j(\phi,\eta) = \prod_{\theta_j=1}^{m-1}(\phi\mathbf{I} + \boldsymbol{\alpha}_{\theta_j}(\eta)), \quad j = 1, 2, \cdots, m.$$

By using the rearrangement (6.37), we can write

$$\mathbf{M}(\phi,\eta) = \sum_{l=0}^{m} \phi^l \mathbf{M}_{0l}(\eta), \tag{6.44}$$

where

$$\mathbf{M}_{0m}(\eta) = \mathbf{I}, \tag{6.45}$$

$$\mathbf{M}_{0(m-k)}(\eta) = \sum_{l_1,l_2,\cdots,l_k \in \mathcal{C}_k^m} \prod_{j=1}^{k} \boldsymbol{\alpha}_{l_j}(\eta), \tag{6.46}$$

where $\mathcal{C}_k^m$ is the set of combinations of k elements out of $\{1, 2, \cdots, m\}$, $k = 1, 2, \cdots, m$. We also can write

$$\widetilde{\mathbf{M}}_j(\phi,\eta) = \sum_{k=0}^{m-1} \phi^k \mathbf{M}_{jk}(\eta), \tag{6.47}$$

where for $j = 1, \cdots, m$, $k = 0, \cdots, m-1$, the matrices $\mathbf{M}_{jk}(\eta)$ are $N \times N$-diagonal matrices whose diagonal elements are defined in the following.

Since the ith diagonal element of $\widetilde{\mathbf{M}}_j(\phi,\eta)$ is $\prod_{\theta_j=1}^{m-1}(\phi + \alpha_{i\theta_j}(\eta))$, we can rewrite this element as $\sum_{k=0}^{m-1} \phi^k c(\eta, i, j, k)$, in which the coefficients $c(\eta, i, j, k)$, according to (6.38) - (6.39), are given by

$$c(\eta, i, j, m-1) = 1, \tag{6.48}$$

$$c(\eta, i, j, m-1-k) = \sum_{l_1,l_2,\cdots,l_k \in \Theta_k^j} \prod_{j=1}^{k} \alpha_{il_j}(\eta), \tag{6.49}$$

where Θ_k^j is the set of combinations of k elements out of $\mathcal{M} \setminus \{j\}$, $k = 1, 2, \cdots, m-1$. For $j \in \mathcal{M}$, $k = 0, \cdots, m-1$, we then can define $\mathbf{M}_{jk}(\eta)$ as the $N \times N$-diagonal matrices with its ith diagonal element equal $c(\eta, i, j, k)$.

From (6.43), (6.44), (6.47), and Lemma 6.3.6 we then obtain for $Re(\phi) \geq 0$, $Re(\eta) > 0$,

$$\begin{aligned}
&\mathbf{Z}^0(\phi,\eta,v)\mathbf{M}(\phi,\eta)\sum_{k=1}^{Nm}\frac{\mathbf{E}^k(\eta)Y_k(\eta)}{(\phi-\gamma_k(1,\eta))}\\
&=\mathbf{A}_0(\eta)\sum_{l=0}^{m}\mathbf{M}_{0l}(\eta)\sum_{k=1}^{Nm}\frac{\phi^l\mathbf{E}^k(\eta)Y_k(\eta)}{(\phi-\gamma_k(1,\eta))}\\
&\quad+\sum_{l=1}^{m}\mathbf{A}_l(\phi,\eta)\sum_{k_1=1}^{m-1}\mathbf{M}_{lk_1}(\eta)\sum_{k_2=1}^{Nm}\frac{\phi^{k_1}\mathbf{E}^{k_2}(\eta)Y_{k_2}(\eta)}{(\phi-\gamma_{k_2}(1,\eta))}\\
&=\mathbf{A}_0(\eta)+\mathbf{A}_0(\eta)\sum_{l=0}^{m}\mathbf{M}_{0l}(\eta)\sum_{k=1}^{Nm}\frac{\gamma_k(1,\eta)^l\mathbf{E}^k(\eta)Y_k(\eta)}{(\phi-\gamma_k(1,\eta))}\\
&\quad+\sum_{l=1}^{m}\mathbf{A}_l(\phi,\eta)\sum_{k_1=1}^{m-1}\mathbf{M}_{lk_1}(\eta)\sum_{k_2=1}^{Nm}\frac{\gamma_{k_2}^{k_1}\mathbf{E}^{k_2}(\eta)Y_{k_2}(\eta)}{(\phi-\gamma_{k_2}(1,\eta))}.
\end{aligned}\tag{6.50}$$

We now rewrite the rational matrix

$$(\phi\tilde{\mathbf{I}}+\widetilde{\boldsymbol{\alpha}}(\eta))^{-1}\mathbf{C}_0\mathbf{M}(\phi,\eta)\sum_{k=1}^{Nm}\frac{\mathbf{E}^k(\eta)Y_k(\eta)}{(\phi-\gamma_k(1,\eta))}$$

which is part of the second term of the right hand-side of (6.42).

By definition, we can write

$$\mathbf{C}_0\mathbf{M}(\phi,\eta)=\begin{pmatrix}\mathbf{I}_{\bar{K}N}\mathbf{M}(\phi,\eta)\\ \mathbf{I}_{\bar{K}N}\mathbf{M}(\phi,\eta)\\ \vdots\\ \mathbf{I}_{\bar{K}N}\mathbf{M}(\phi,\eta)\end{pmatrix},$$

so that by rearrangement (6.37),

$$\begin{aligned}
(\phi\tilde{\mathbf{I}}+\widetilde{\boldsymbol{\alpha}}(\eta))^{-1}\mathbf{C}_0\mathbf{M}(\phi,\eta)&=\begin{pmatrix}\mathbf{I}_{\bar{K}N}(\phi\mathbf{I}+\boldsymbol{\alpha}_1)^{-1}\mathbf{M}(\phi,\eta)\\ \mathbf{I}_{\bar{K}N}(\phi\mathbf{I}+\boldsymbol{\alpha}_2)^{-1}\mathbf{M}(\phi,\eta)\\ \vdots\\ \mathbf{I}_{\bar{K}N}(\phi\mathbf{I}+\boldsymbol{\alpha}_m)^{-1}\mathbf{M}(\phi,\eta)\end{pmatrix}\\
&=\begin{pmatrix}\mathbf{I}_{\bar{K}N}\widetilde{\mathbf{M}}_1(\eta)\\ \mathbf{I}_{\bar{K}N}\widetilde{\mathbf{M}}_2(\eta)\\ \vdots\\ \mathbf{I}_{\bar{K}N}\widetilde{\mathbf{M}}_m(\eta)\end{pmatrix}\\
&=\begin{pmatrix}\mathbf{I}_{\bar{K}N}\sum_{k=0}^{m-1}\mathbf{M}_{1k}(\eta)\\ \mathbf{I}_{\bar{K}N}\sum_{k=0}^{m-1}\mathbf{M}_{2k}(\eta)\\ \vdots\\ \mathbf{I}_{\bar{K}N}\sum_{k=0}^{m-1}\mathbf{M}_{mk}(\eta)\end{pmatrix}.
\end{aligned}\tag{6.51}$$

Define $\bar{K}m \times N$-dimensional matrices:

$$\mathbf{L}_{m-1}(\eta) = \begin{pmatrix} \mathbf{I}_{\bar{K}N} \\ \mathbf{I}_{\bar{K}N} \\ \vdots \\ \mathbf{I}_{\bar{K}N} \end{pmatrix} = \mathbf{C}_0, \tag{6.52}$$

$$\mathbf{L}_l(\eta) = \begin{pmatrix} \mathbf{I}_{\bar{K}N}\mathbf{M}_{1l}(\eta) \\ \mathbf{I}_{\bar{K}N}\mathbf{M}_{2l}(\eta) \\ \vdots \\ \mathbf{I}_{\bar{K}N}\mathbf{M}_{ml}(\eta) \end{pmatrix}, \quad l = 0, 1, \cdots, m-2, \tag{6.53}$$

where the matrices $\mathbf{M}_{jl}(\eta)$ are defined on page 153. With the definition of $\mathbf{L}_l(\eta)$, $l = 0, \cdots, m-1$ above, we obtain the following lemma.

Lemma 6.3.7
For $Re(\phi) \geq 0$, $Re(\eta) > 0$,

$$(\phi\tilde{\mathbf{I}} + \widetilde{\boldsymbol{\alpha}}(\eta))^{-1}\mathbf{C}_0\mathbf{M}(\phi,\eta)\sum_{k=1}^{Nm}\frac{\mathbf{E}^k(\eta)Y_k(\eta)}{(\phi-\gamma_k(1,\eta))} = \sum_{l=0}^{m-1}\mathbf{L}_l(\eta)\sum_{k=1}^{Nm}\frac{\gamma_k(1,\eta)^l\mathbf{E}^k(\eta)Y_k(\eta)}{(\phi-\gamma_k(1,\eta))}. \tag{6.54}$$

Proof. From (6.51) we have

$$(\phi\tilde{\mathbf{I}} + \widetilde{\boldsymbol{\alpha}}(\eta))^{-1}\mathbf{C}_0\mathbf{M}(\phi,\eta)\sum_{k=1}^{Nm}\frac{\mathbf{E}^k(\eta)Y_k(\eta)}{(\phi-\gamma_k(1,\eta))} = \sum_{l=0}^{m-1}\mathbf{L}_l(\eta)\sum_{k=1}^{Nm}\frac{\phi^l\mathbf{E}^k(\eta)Y_k(\eta)}{(\phi-\gamma_k(1,\eta))}.$$

By using Lemma 6.3.6, we then obtain (6.54). ■

From (6.75), (6.50), and (6.51) we obtain an explicit expression for $\mathbf{Z}(1,\phi,\eta,v)$, which we write in the following theorem.

Theorem 6.3.3
If Conditions 6.3.1 and 6.3.2 are satisfied then for $Re(\phi) \geq 0, Re(\eta) > 0$,

$$\begin{aligned}\mathbf{Z}(1,\phi,\eta,v) &= \mathbf{A}_0(\eta) + \mathbf{A}_0(\eta)\sum_{l=0}^{m}\mathbf{M}_{0l}(\eta)\sum_{k=1}^{Nm}\frac{\gamma_k(1,\eta)^l\mathbf{E}^k(\eta)Y_k(\eta)}{(\phi-\gamma_k(1,\eta))} \\ &\quad + \sum_{l=1}^{m}\mathbf{A}_l(\phi,\eta)\sum_{k_1=0}^{m-1}\mathbf{M}_{lk_1}(\eta)\sum_{k_2=1}^{Nm}\frac{\gamma_{k_2}^{k_1}\mathbf{E}^{k_2}(\eta)Y_{k_2}(\eta)}{(\phi-\gamma_{k_2}(1,\eta))} \\ &\quad - \mathbf{A}(\eta)\mathbf{C}_0 - \mathbf{A}(\eta)\sum_{l=0}^{m-1}\mathbf{L}_l(\eta)\sum_{k=1}^{Nm}\frac{\gamma_k(1,\eta)^{l+1}\mathbf{E}^k(\eta)Y_k(\eta)}{(\phi-\gamma_k(1,\eta))},\end{aligned} \tag{6.55}$$

where

$$\mathbf{A}(\eta) = \sum_{i=1}^{\bar{K}m}\frac{\mathbf{Z}^0(\gamma_i(1,\eta),\eta,v)\mathbf{D}^i(1,\eta)(\mathbf{S}(1,\eta)^{-1})_i}{\gamma_i(1,\eta)}.$$

With this theorem, it is easy to obtain closed-form Laplace-Stieltjes transforms of the probability distributions of interest, which will be derived in the sections 6.4 and 6.5.

6.4 Inverse transformation

In this section we consider the distribution functions $F_{ij}(x,t,v)$, for $i,j \in \mathcal{N}$, of the buffer content at time $t \geq 0$ for initial buffer content $V_0 = v$.

If $J_t = j$ then V_t, the buffer content at time t, satisfies the relation

$$V_t = [W_{N_t} + r_j(t - T_{N_t})]^+,$$

where N_t is the number of transitions of the process $\{J_t\}$ during $(0,t]$. Consequently, for $Re(\phi) \geq 0$,

$$\begin{aligned} & E\left(e^{-\phi V_t}\mathbf{1}(J_t = j)|X_1 = i, V_0 = v\right) \\ =& E\left(e^{-\phi[W_0 + r_j(t-T_0)]^+}\mathbf{1}(T_0 \leq t < T_1, J_t = j)|X_1 = i, V_0 = v\right) \\ & + \sum_{n=1}^{\infty} E\left(e^{-\phi[W_n + r_j(t-T_n)]^+}\mathbf{1}(T_n \leq t < T_{n+1}, J_t = j)|X_1 = i, V_0 = v\right), \end{aligned} \tag{6.56}$$

where

$$\begin{aligned} & \sum_{n=1}^{\infty} E\left(e^{-\phi[W_n + r_j(t-T_n)]^+}\mathbf{1}(T_n \leq t < T_{n+1}, J_t = j)|X_1 = i, V_0 = v\right) \\ =& \sum_{n=1}^{\infty}\sum_{l=1}^{N} E\left(e^{-\phi[W_n + r_j(t-T_n)]^+}\mathbf{1}(T_n \leq t < T_{n+1}, X_n = l, X_{n+1} = j)\right. \\ & \quad |X_1 = i, V_0 = v) \\ =& \sum_{n=1}^{\infty}\sum_{l=1}^{N} \int_0^t P(A_{n+1} > t - u, X_{n+1} = j|X_n = l) \\ & \quad .d_u E\left(e^{-\phi[W_n + r_j(t-T_n)]^+}\mathbf{1}(T_n \leq u, X_n = l)|X_1 = i, V_0 = v\right) \\ =& \sum_{n=1}^{\infty}\sum_{l=1}^{N} \int_0^t P_{lj}\sum_{k=1}^{m} a_{jk}e^{-\mu_{jk}(t-u)} \\ & \quad .d_u E\left(e^{-\phi[W_n + r_j(t-T_n)]^+}\mathbf{1}(T_n \leq u, X_n = l)|X_1 = i, V_0 = v\right). \end{aligned} \tag{6.57}$$

From the identity(see page 269 of [17])

$$e^{-\phi x^+} = \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\frac{\phi}{\phi - \xi}e^{-\xi x}, \text{ with } x \text{ real}, \ Re(\phi) > Re(\xi) > 0,$$

where the path of integration is a line parallel to the imaginary axis, we have

$$\begin{aligned} & E\left(e^{-\phi[W_n + r_j(t-T_n)]^+}\mathbf{1}(T_n \leq u, X_n = l)|X_1 = i, V_0 = v\right) \\ =& \frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi}\frac{\phi}{\phi - \xi}e^{-\xi r_j t} E\left(e^{-\xi W_n + r_j \xi T_n}\mathbf{1}(T_n \leq u, X_n = l)\right. \\ & \quad |X_1 = i, V_0 = v) \end{aligned} \tag{6.58}$$

and

$$\begin{aligned}&E\left(e^{-\phi[W_0+r_j(t-T_0)]^+}\mathbf{1}(T_0\le t<T_1,J_t=j)|X_1=i,V_0=v\right)\\=&\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}\frac{\phi}{\phi-\xi}e^{-\xi(v+r_jt)}E\left(\mathbf{1}(t<T_1,J_t=j)|X_1=i,V_0=v\right).\end{aligned}\tag{6.59}$$

Combining (6.56), (6.57),(6.58) and (6.59) yields

$$\begin{aligned}&\int_0^\infty e^{-\eta t}E\left(e^{-\phi V_t}\mathbf{1}(J_t=j)|X_1=i,V_0=v\right)dt\\
=&\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}\frac{\phi}{\phi-\xi}e^{-\xi v}\int_0^\infty e^{-\eta t-\xi r_jt}P\left(t<T_1,J_t=j|X_1=i,V_0=v\right)dt\\
&+\sum_{n=1}^\infty\sum_{l=1}^N\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}\frac{\phi}{\phi-\xi}\int_0^\infty e^{-\eta t}\int_0^t P_{lj}\sum_{k=1}^m a_{jk}e^{-\mu_{jk}(t-u)}e^{-\xi r_jt}\\
&.d_uE\left(e^{-\xi W_n+r_j\xi T_n}\mathbf{1}(T_n\le u,X_n=l)|X_1=i,V_0=v\right)dt\\
=&\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}\frac{\phi}{\phi-\xi}e^{-\xi v}\int_0^\infty e^{-\eta t-\xi r_jt}\sum_{k=1}^m a_{jk}e^{-\mu_{jk}t}\delta_{ij}dt\\
&+\sum_{n=1}^\infty\sum_{l=1}^N\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}\frac{\phi}{\phi-\xi}E\left(e^{-\xi W_n-\eta T_n}\mathbf{1}(X_n=l)|X_1=i,V_0=v\right)\\
&\qquad .P_{lj}\sum_{k=1}^m a_{jk}/(\eta+\mu_{jk}+\xi r_j)\\
=&\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}\frac{\phi}{\phi-\xi}e^{-\xi v}\delta_{ij}\sum_{k=1}^m\frac{a_{jk}}{(r_j\xi+\mu_{jk}+\eta)}\\
&+\sum_{l=1}^N\frac{1}{2\pi i}\int_{-i\infty+0}^{i\infty+0}\frac{d\xi}{\xi}\frac{\phi}{\phi-\xi}Z_{il}(1,\xi,\eta,v)P_{lj}\sum_{k=1}^m\frac{a_{jk}}{(\eta+\mu_{jk}+\xi r_j)}.\end{aligned}\tag{6.60}$$

For $Re(\phi)\ge 0, Re(\eta)>0$, let

$$Z^*_{ij}(\phi,\eta,v)=\int_0^\infty e^{-\eta t}E\left(e^{-\phi V_t}\mathbf{1}(J_t=j)|X_1=i,V_0=v\right)dt.\tag{6.61}$$

We evaluate the integrals in (6.60) by considering the poles of the integrands in the right half-plane $Re(\phi)>0$. It follows that for $Re(\phi)\ge 0$, $Re(\eta)>0$, $j\in R^-$,

$$\begin{aligned}Z^*_{ij}(\phi,\eta,v)=&\frac{\delta_{ij}}{r_j}\sum_{k=1}^m\frac{a_{jk}\left[\alpha_{jk}(\eta)e^{-\phi v}+\phi e^{\alpha_{jk}(\eta)v}\right]}{\alpha_{jk}(\eta)(\phi+\alpha_{jk}(\eta))}\\
&+\frac{1}{r_j}\sum_{l=1}^N Z_{il}(1,\phi,\eta,v)P_{lj}\sum_{k=1}^m\frac{a_{jk}}{(\phi+\alpha_{jk}(\eta))}\\
&+\frac{1}{r_j}\sum_{l=1}^N\sum_{k=1}^m Z_{il}(1,-\alpha_{jk}(\eta),\eta,v)P_{lj}\frac{\phi}{(\phi+\alpha_{jk}(\eta))}\frac{a_{jk}}{\alpha_{jk}(\eta)},\end{aligned}\tag{6.62}$$

and for $j \in R^+$,

$$\begin{aligned} Z^*_{ij}(\phi,\eta,v) =& \frac{1}{r_j} e^{-\phi v} \delta_{ij} \sum_{k=1}^{m} \frac{a_{jk}}{(\phi+\alpha_{jk}(\eta))} \\ &+ \frac{1}{r_j} \sum_{l=1}^{N} Z_{il}(1,\phi,\eta,v) P_{lj} \sum_{k=1}^{m} \frac{a_{jk}}{(\phi+\alpha_{jk}(\eta))}. \end{aligned} \tag{6.63}$$

It is easy to see that for $k = 1, 2, \cdots, m$, $-\alpha_{jk}(\eta)$ is not a pole of $Z^*_{ij}(\phi,\eta,v)$ as a function of ϕ in the domain $Re(\phi) > 0$.

By substituting the explicit expression for $Z_{il}(1,\phi,\eta,v)$ given by (6.55), we can invert $Z^*_{ij}(\phi,\eta,v)$ analytically to obtain the Laplace-Stieltjes transform

$$\xi_{ij}(x,\eta,v) = \int_0^\infty e^{-\eta t} F_{ij}(x,t,v)dt, \quad Re(\eta) \geq 0, i,j \in \mathcal{N}. \tag{6.64}$$

The expressions for $\xi_{i,j}(x,\eta,v), i,j \in \mathcal{N}$ given in Theorem 6.4.1 below are obtained by contour integration through the inversion formula for Laplace-Stieltjes transform, see Lemma A.3.1 in the appendix. Since the only singularities of $Z^*_{ij}(\phi,\eta,v)$ in (6.62) and (6.63) are simple poles, the result of this contour integration is obtained from the corresponding residues.

From (6.55) and (A.10) we see that the poles of $Z_{ij}(1,\phi,\eta,v)$ in the left half-plane $Re(\phi) < 0$ are $\gamma_l(1,\eta)$, $l = \bar{K}m+1, \bar{K}m+2, \cdots, Nm$. The $-\alpha_{ik}(\eta), k = 1, 2, \cdots, m$, which are the poles of $Z^0_i(\phi,\eta,v)$ in the left half-plane $Re(\phi < 0)$ for $i \in R^+$, are not poles of $Z_{ij}(1,\phi,\eta,v)$ since the factor $1/\prod_{k=1}^m(\phi+\alpha_{ik}(\eta))$ of $Z^0_i(\phi,\eta,v)$ in $Z_{ij}(1,\phi,\eta,v)$ will be cancelled when multiplying $\mathbf{Z}^0(\phi,\eta,v)$ by $\mathbf{M}(\phi,\eta)$.

From (6.62) and (6.63), it can be seen that for $j \in R^-$, the poles of $Z^*_{ij}(1,\phi,\eta,v)$ are the poles of $Z_{il}(1,\phi,\eta,v), l = 1, 2, \cdots, N$, and for $j \in R^+$, the poles of $Z^*_{ij}(\phi,\eta,v)$ are the poles of $Z_{il}(1,\phi,\eta,v), l = 1, 2, \cdots, N$, and $\phi = -\alpha_{jk}(\eta)$, $k = 1, 2, \cdots, m$.

Define for $l_1 = \bar{K}m+1, \cdots, Nm$, the $N \times N$-matrix

$$U^{l_1}_{il}(\eta) = \left[\mathbf{A}(\eta) \sum_{l_2=0}^{m-1} \mathbf{L}_{l_2}(\eta) \gamma^{l_2}_{l_1} \mathbf{E}^{l_1}(\eta) \mathbf{Y}_{l_1}(\eta) \right]_{il}. \tag{6.65}$$

We denote by $\mathbf{B}^k_j(\eta)$ the kth diagonal element of the matrix $\mathbf{B}_j(\eta)$. Then by using all poles of $Z^*_{ij}(\phi,\eta,v)$, and by using the expression for $Z^0_i(\phi,\eta,v)$ given in Theorem 6.2.1, we obtain after length but straightforward calculations, taking into account the above arguments, the following theorem.

Theorem 6.4.1
If Condition 6.3.1 and 6.3.2 are satisfied, then for $x \geq 0$, $Re(\eta) > 0$,

$$
\begin{aligned}
&\xi_{ij}(x,\eta,v) \\
&= -\frac{1}{r_j}\sum_{l_1=\bar{K}m+1}^{Nm}\sum_{l_2=1}^{N}\mathbf{A}_0^i(\eta)\sum_{l_3=0}^{m}\mathbf{M}_{0l_3}^i(\eta)\gamma_{l_1}^{l_3-1}(1,\eta)\left(\mathbf{E}^{l_1}(\eta)\mathbf{Y}_{l_1}(\eta)\right)_{il_2}P_{l_2j} \\
&\quad .\sum_{k=1}^{m}\frac{a_{jk}}{(\gamma_{l_1}(1,\eta)+\alpha_{jk}(\eta))}\left(1-e^{\gamma_{l_1}x}\right) \\
&\quad -\frac{1}{r_j}\sum_{l_1=\bar{K}m+1}^{Nm}\sum_{l_2=1}^{N}\sum_{l_3=1}^{m}\frac{a_{il_3}\mu_{il_3}}{r_i}\sum_{l_4=0}^{m-1}\mathbf{M}_{l_3l_4}^i(\eta)\gamma_{l_1}^{l_4-1}(1,\eta)\left(\mathbf{E}^{l_1}(\eta)\mathbf{Y}_{l_1}(\eta)\right)_{il_2} \\
&\quad .P_{l_2j}\sum_{k=1}^{m}\frac{a_{jk}}{(\gamma_{l_1}(1,\eta)+\alpha_{jk}(\eta))}\left(1-e^{\gamma_{l_1}(x-v)}\right)\mathcal{H}(x-v) \\
&\quad +\frac{1}{r_j}\sum_{l_1=\bar{K}m+1}^{Nm}\sum_{l_2=1}^{N}\sum_{l_3=1}^{m}\frac{a_{il_3}\mu_{il_3}e^{\alpha_{il_3}(\eta)}}{r_i}\sum_{l_4=0}^{m-1}\mathbf{M}_{l_3l_4}^i(\eta)\gamma_{l_1}^{l_4-1}(1,\eta)\left(\mathbf{E}^{l_1}(\eta)\mathbf{Y}_{l_1}(\eta)\right)_{il_2} \\
&\quad .P_{l_2j}\sum_{k=1}^{m}\frac{a_{jk}}{(\gamma_{l_1}(1,\eta)+\alpha_{jk}(\eta))}\left(1-e^{\gamma_{l_1}x}\right) \\
&\quad +\frac{1}{r_j}\sum_{l_1=\bar{K}m+1}^{Nm}\sum_{l_2=1}^{N}U_{il_2}^{l_1}(\eta,v)P_{l_2j}\sum_{k=1}^{m}\frac{a_{jk}}{(\gamma_{l_1}(1,\eta)+\alpha_{jk}(\eta))}\left(1-e^{\gamma_{l_1}(1,\eta)x}\right),
\end{aligned}
\tag{6.66}
$$

for $i,j \in R^-$,

$$
\begin{aligned}
&\xi_{ij}(x,\eta,v) \\
&= -\frac{1}{r_j}\sum_{l_1=\bar{K}m+1}^{Nm}\sum_{l_2=1}^{N}\sum_{l_3=1}^{m}\mathbf{A}_{l_3}^i(\eta)\sum_{l_4=1}^{m-1}\mathbf{M}_{l_3l_4}^i(\eta)\gamma_{l_1}^{l_4-1}(1,\eta)\left(\mathbf{E}^{l_1}(\eta)\mathbf{Y}_{l_1}(\eta)\right)_{il_2} \\
&\quad .P_{l_2j}\sum_{k=1}^{m}\frac{a_{jk}}{(\gamma_{l_1}(1,\eta)+\alpha_{jk}(\eta))}\left(1-e^{\gamma_{l_1}(x-v)}\right)\mathcal{H}(x-v) \\
&\quad +\frac{1}{r_j}\sum_{l_1=\bar{K}m+1}^{Nm}\sum_{l_2=1}^{N}U_{il_2}^{l_1}(\eta,v)P_{l_2j}\sum_{k=1}^{m}\frac{a_{jk}}{(\gamma_{l_1}(1,\eta)+\alpha_{jk}(\eta))}\left(1-e^{\gamma_{l_1}(1,\eta)x}\right),
\end{aligned}
\tag{6.67}
$$

for $i \in R^+$ and $j \in R^-$,

$$\begin{aligned}
&\xi_{ij}(x,\eta,v)\\
=&\frac{\delta_{ij}}{r_j}\sum_{k=1}^{m}\frac{a_{jk}}{\alpha_{jk}(\eta)}\left(1-e^{-\alpha_{jk}(\eta)(x-v)}\right)\mathcal{H}(x-v)\\
&-\frac{1}{r_j}\sum_{l_1=\bar{K}m+1}^{Nm}\sum_{l_2=1}^{N}\mathbf{A}_0^i(\eta)\sum_{l_3=0}^{m}\mathbf{M}_{0l_3}^i(\eta)\gamma_{l_1}^{l_3-1}(1,\eta)\left(\mathbf{E}^{l_1}(\eta)\mathbf{Y}_{l_1}(\eta)\right)_{il_2}P_{l_2j}\\
&.\sum_{k=1}^{m}\frac{a_{jk}}{(\gamma_{l_1}(1,\eta)+\alpha_{jk}(\eta))}\left(1-e^{\gamma_{l_1}x}\right)\\
&-\frac{1}{r_j}\sum_{l_1=\bar{K}m+1}^{Nm}\sum_{l_2=1}^{N}\sum_{l_3=1}^{m}\frac{a_{il_3}\mu_{il_3}}{r_i}\sum_{l_4=0}^{m-1}\mathbf{M}_{l_3l_4}^i(\eta)\gamma_{l_1}^{l_4-1}(1,\eta)\left(\mathbf{E}^{l_1}(\eta)\mathbf{Y}_{l_1}(\eta)\right)_{il_2}\\
&.P_{l_2j}\sum_{k=1}^{m}\frac{a_{jk}}{(\gamma_{l_1}(1,\eta)+\alpha_{jk}(\eta))}\left(1-e^{\gamma_{l_1}(x-v)}\right)\mathcal{H}(x-v)\\
&-\frac{1}{r_j}\sum_{l_1=\bar{K}m+1}^{Nm}\sum_{l_2=1}^{N}\sum_{l_3=1}^{m}\frac{a_{il_3}\mu_{il_3}e^{\alpha_{il_3}(\eta)}}{r_i}\sum_{l_4=0}^{m-1}\mathbf{M}_{l_3l_4}^i(\eta)\gamma_{l_1}^{l_4-1}(1,\eta)\left(\mathbf{E}^{l_1}(\eta)\mathbf{Y}_{l_1}(\eta)\right)_{il_2}\\
&.P_{l_2j}\sum_{k=1}^{m}\frac{a_{jk}}{(\gamma_{l_1}(1,\eta)+\alpha_{jk}(\eta))}\left(1-e^{\gamma_{l_1}x}\right)\\
&+\frac{1}{r_j}\sum_{l_1=\bar{K}m+1}^{Nm}\sum_{l_2=1}^{N}U_{il_2}^{l_1}(\eta,v)P_{l_1j}\sum_{k=1}^{m}\frac{a_{jk}}{(\gamma_{l_1}(1,\eta)+\alpha_{jk}(\eta))}\left(1-e^{\gamma_{l_1}(1,\eta)x}\right),
\end{aligned}\tag{6.68}$$

for $i \in R^-$ and $j \in R^+$,

$$\begin{aligned}
&\xi_{ij}(x,\eta,v)\\
=&\frac{\delta_{ij}}{r_j}\sum_{k=1}^{m}\frac{a_{jk}}{\alpha_{jk}(\eta)}\left(1-e^{-\alpha_{jk}(\eta)(x-v)}\right)\mathcal{H}(x-v)\\
&-\frac{1}{r_j}\sum_{l_1=\bar{K}m+1}^{Nm}\sum_{l_2=1}^{N}\sum_{l_3=1}^{m}\mathbf{A}_{l_3}^i(\eta)\sum_{l_4=1}^{m-1}\mathbf{M}_{l_3l_4}^i(\eta)\gamma_{l_1}^{l_4-1}(1,\eta)\left(\mathbf{E}^{l_1}(\eta)\mathbf{Y}_{l_1}(\eta)\right)_{il_2}\\
&.P_{l_2j}\sum_{k=1}^{m}\frac{a_{jk}}{(\gamma_{l_1}(1,\eta)+\alpha_{jk}(\eta))}\left(1-e^{\gamma_{l_1}(x-v)}\right)\mathcal{H}(x-v)\\
&+\frac{1}{r_j}\sum_{l_1=\bar{K}m+1}^{Nm}\sum_{l_2=1}^{N}U_{il_2}^{l_1}(\eta,v)P_{l_1j}\sum_{k=1}^{m}\frac{a_{jk}}{(\gamma_{l_1}(1,\eta)+\alpha_{jk}(\eta))}\left(1-e^{\gamma_{l_1}(1,\eta)x}\right).
\end{aligned}\tag{6.69}$$

for $i, j \in R^+$.

To get the distribution functions

$$F_{ij}(x,t,v), \quad i,j \in \mathcal{N},$$

we can do a numerical inversion with respect to η of (6.66) - (6.69).

6.5 The steady-state distribution of the buffer content

We have derived an explicit expression for the Laplace-Stieltjes transform $\xi_{ij}(x,\ \eta, v)$ in Theorem 6.4.1. In this section we will use Abel's limit theorem to get the steady-state distribution of the buffer content if it exists. It is clear that the steady-state distribution of the buffer content in continuous time exists if it exists at transition epochs. The following theorem states the necessary and sufficient condition for the existence of the limiting distribution of the buffer content at transition epochs.

Theorem 6.5.1
The process $\{(W_n, V_n), n = 0, 1, \cdots\}$ *weakly converges to* (W, V) *if and only if*

$$\sum_{j=1}^{N} \pi_j r_j < 0.$$

Proof. The process $\{(W_n, X_n)\}$ is regenerative where for any $i \in \mathcal{N}$ the state $(0, i)$ can be seen as the regenerative state. Since the process $\{X_n\}$ is assumed to have a limiting distribution, the return times of the process $\{(W_n, X_n)\}$ are aperiodic so that $\lim_{n\to\infty} P\{W_n \leq x, X_n = j|X_1 = i, V_0 = v\}$ for $x \geq 0$ exists. If this limit is zero then no limiting distribution exists, otherwise (W_n, X_n) converges weakly to a stationary random vector (W, X).

From (6.26), (6.27) and Lemma 6.3.2 we see that if and only if $\sum_{i=1}^{N} \pi_i r_i < 1$, for $\phi \neq 0$,

$$\begin{aligned}
&\lim_{z\uparrow 1} (1-z)K_{ij}^{+}(z,\phi,0,v) + (1-z)K_{ij}^{-}(z,0,0,v) \\
=&\lim_{z\uparrow 1}(1-z)\delta_{ij}Z_i^0(\phi,0,v) - \lim_{z\uparrow 1}(1-z)\sum_{k=1}^{\bar{K}m} D_{ik}(z,0)\frac{Z_i^0(\gamma_k(z,0),0,v)}{\gamma_k(z,0)}C_{kj}(z,0) \\
&+\lim_{z\uparrow 1}(1-z)\sum_{k=1}^{\bar{K}m} D_{ik}(z,0)\frac{Z_i^0(\phi,0,v) - Z_i^0(\gamma_k(z,0),0,v)}{\phi - \gamma_k(z,0)}C_{kj}(z,0) \qquad (6.70)\\
=&-D_{i1}(1,0)Z_i^0(\gamma_1(1,0),0,v)\lim_{z\uparrow 1}\frac{(1-z)}{\gamma_1(z,0)}C_{1j}(1,0) \\
=&-D_{i1}(1,0)\lim_{z\uparrow 1}\frac{(1-z)}{\gamma_1(z,0)}C_{1j}(1,0),
\end{aligned}$$

where

$$\lim_{z\uparrow 1}\frac{(1-z)}{\gamma_1(z,0)} = -\frac{1}{\left(\sum_{j=1}^{N}\pi_j r_j\right)\left(\sum_{j=1}^{N} p_j \sum_{k=1}^{m}\frac{a_{jk}}{\mu_{jk}}\right)} \neq 0,$$

so that

$$\lim_{n\to\infty} E\left(e^{-\phi W_n}\mathbf{1}(X_n = j)|X_1 = i, W_0 = v\right) = \lim_{z\uparrow 1}(1-z)Z_{ij}(z,\phi,0,v) \neq 0.$$

It follows that $\lim_{n\to\infty} P\{W_n \le x, X_n = j | X_1 = i, V_0 = v\} \neq 0$. We then can conclude that if $\sum_{j=1}^N \pi_j r_j < 0$, the process (W_n, X_n) converges weakly to a stationary random vector (W, X).

■

In the following subsection, we will derive the distribution function $P(W \le x, X = j)$, which from Theorem 6.5.1 exists if and only if $\sum_{j=1}^N \pi_j r_j < 0$. Then in subsection 6.5.2 we will derive the distribution function of the steady-state buffer content in continuous time based on the result in subsection 6.5.1.

6.5.1 The steady-state distribution of buffer content at transition epochs

Let

$$\mathbf{E}^i = \mathbf{E}^i(1,0), \text{ and } \gamma_i = \gamma_i(1,0), i = 1, 2, \cdots, Nm.$$

By Lemma 6.3.2 we know that $\gamma_1 = 0$. Let $\mathbf{Y} = \mathbf{Y}(0)$, $\mathbf{D} = \mathbf{D}(1,0)$ and $\mathbf{C} = \mathbf{C}(1,0)$. Since $\mathbf{H}(1,0,0) = \mathbf{I} - \mathbf{G}(0,0) = \mathbf{I} - \mathbf{P}$ and $\mathbf{H}(1,0,0)\mathbf{D}^1(1,0) = 0$, we may put $\mathbf{D}^1(1,0) = \mathbf{1}$. It follows that $E_{i1} = \frac{1}{\prod_{j=1}^m \alpha_{ij}}$,

$$\mathbf{S}^1(1,0) = (\frac{1}{\alpha_{11}}, \frac{1}{\alpha_{21}}, \cdots, \frac{1}{\alpha_{\bar{K}1}}, \frac{1}{\alpha_{12}}, \frac{1}{\alpha_{22}}, \cdots, \frac{1}{\alpha_{\bar{K}2}}, \cdots, \frac{1}{\alpha_{1m}}, \cdots, \frac{1}{\alpha_{\bar{K}m}})^t.$$

Let

$$\mathcal{V}_1 = \frac{1}{\frac{d}{dz}\gamma_1(z,0)\big|_{z=1}} = \sum_{j=1}^N p_j r_j \sum_{k=1}^m \frac{a_{jk}}{\mu_{jk}}.$$

We also may put

$$\mathbf{Y}_1 = \frac{1}{\mathcal{V}_1}\mathbf{p},$$

where $\mathbf{p}$ is the stationary probability distribution vector of the Markov chain $\{X_n\}$.

Let for $Re(\phi) \ge 0$,

$$\mathbf{H}(\phi) = \mathbf{H}(1, \phi, 0),$$

$$\tilde{\mathbf{K}}(\phi) = \mathbf{I} + \mathbf{D}\text{diag}\left(0, \frac{1}{\phi - \gamma_2}, \ldots, \frac{1}{\phi - \gamma_{Nm}}\right)\mathbf{C}.$$

Moreover, let for $Re(\phi) \ge 0$,

$$\begin{aligned}
\mathbf{X}(\phi) &= \mathbf{X}(\phi, 0), \\
\mathbf{M}(\phi) &= \mathbf{M}(\phi, 0), \\
\mathbf{S} &= \mathbf{S}(1, 0), \\
\mathbf{S}^{-1} &= \mathbf{S}(1, 0)^{-1},
\end{aligned}$$

and

$$\mathbf{H}^+(\phi) = \begin{cases} \mathbf{H}(\phi)\tilde{\mathbf{K}}(\phi) + \frac{1}{\phi}\mathbf{H}(\phi)\mathbf{D}^1\mathbf{C}_1, & \text{if } \phi \neq 0, \\ \\ \mathbf{H}(0)\tilde{\mathbf{K}}(0) + \mathbf{H}'(0)\mathbf{D}^1\mathbf{C}_1, & \text{if } \phi = 0. \end{cases}$$

The matrix $\mathbf{H}^+(\phi)$ is the limit of $\mathbf{H}^+(z, \phi, 0)$ for $z \uparrow 1$. Let $\mathbf{Z}(\phi)$ be the $N \times N-$dimensional matrix with elements

$$\begin{aligned} Z_{ij}(\phi) &= E\left(e^{-\phi W}\mathbf{1}(X = j)|X_1 = i, W_0 = v\right) \\ &= \lim_{z\uparrow 1}(1 - z)Z_{ij}(z, \phi, 0, v), j = 1, 2, \ldots, N. \end{aligned}$$

From Theorem 6.3.2 and (6.70) we now have for $Re(\phi) \geq 0$,

$$\begin{aligned} \mathbf{Z}(\phi)\mathbf{H}^+(\phi) &= \frac{1}{\frac{d}{dz}\gamma_1(z, 0)\big|_{z=1}}\mathbf{D}^1\mathbf{C}_1 \\ &= \mathcal{V}_1\mathbf{U}, \end{aligned} \tag{6.71}$$

where $\mathbf{U}$ is an $N \times N-$dimensional matrix with rows $\mathbf{C}_1$. This shows that $Z_{ij}(\phi)$ does not depend on i and v.

From (6.71), if Conditions 6.3.1 and 6.3.2 are satisfied, we can find a closed form expression for the Laplace-Stieltjes transform of the probability distribution in steady state of the buffer content at transition epochs.

Let $\bar{\mathbf{Z}}(\phi)$ be the $N-$ dimensional row vector with components $Z_j(\phi), j = 1, 2, \cdots, N$. From (6.71) we obtain for $Re(\phi) \geq 0$, using Lemma 6.3.1, Lemma 6.3.2, and Theorem 6.3.1, and (6.34),

$$\begin{aligned} \bar{\mathbf{Z}}(\phi) &= \mathcal{V}_1\mathbf{C}_1\mathbf{H}^+(\phi)^{-1} \\ &= \mathcal{V}_1\mathbf{C}_1\left[\mathbf{I} - \mathbf{D}\mathbf{X}(\phi)^{-1}\mathbf{C}\right]\mathbf{M}(\phi)\sum_{k=1}^{Nm}\frac{\mathbf{E}^k\mathbf{Y}_k}{(\phi - \gamma_k)} \end{aligned} \tag{6.72}$$

We can show, in a similar way as for the derivation of (6.41), that for $Re(\phi) \geq 0$,

$$\mathbf{C}_1\left[\mathbf{I} - \mathbf{D}\mathbf{X}(\phi)^{-1}\mathbf{C}\right] = \phi\mathbf{1}_1\mathbf{S}^{-1}\left(\phi\tilde{\mathbf{I}} + \tilde{\boldsymbol{\alpha}}(0)\right)^{-1}\mathbf{C}_0 = \phi\mathbf{S}_1^{-1}\left(\phi\tilde{\mathbf{I}} + \tilde{\boldsymbol{\alpha}}(0)\right)^{-1}\mathbf{C}_0. \tag{6.73}$$

By substituting (6.73) into (6.72), and by letting

$$\mathcal{V}_2 = \mathcal{V}_1\mathbf{S}_1^{-1},$$

we obtain for $Re(\phi) \geq 0$,

$$\bar{\mathbf{Z}}(\phi) = \mathcal{V}_2\phi\left(\phi\mathbf{I} + \widetilde{\boldsymbol{\alpha}}(0)\right)^{-1}\mathbf{C}_0\mathbf{M}(\phi)\sum_{k=1}^{Nm}\frac{\mathbf{E}^k\mathbf{Y}_k}{(\phi - \gamma_k)}. \tag{6.74}$$

By using Lemma 6.3.7, and setting $\eta = 0$, we then obtain for $Re(\phi) \geq 0$,

$$\bar{\mathbf{Z}}(\phi) = \mathcal{V}_2\mathbf{L}_0(0)\sum_{k=1}^{Nm}\frac{\phi\mathbf{E}^k\mathbf{Y}_k}{(\phi - \gamma_k)} + \sum_{l=1}^{m-1}\mathbf{L}_l(0)\sum_{k=1}^{Nm}\frac{\phi\gamma_k^l\mathbf{E}^k\mathbf{Y}_k}{(\phi - \gamma_k)}. \tag{6.75}$$

Noting that

$$\mathcal{V}_2\mathbf{L}_0(0)\mathbf{E}^1\mathbf{Y}_1 = \mathcal{V}_1\mathbf{Y}_1 = \mathbf{p}$$

and

$$\frac{\phi\gamma_k^l \mathbf{E}^k \mathbf{Y}_k}{(\phi-\gamma_k)} = \gamma_k^l \mathbf{E}^k \mathbf{Y}_k + \frac{\gamma_k^{l+1} \mathbf{E}^k \mathbf{Y}_k}{(\phi-\gamma_k)},$$

where $\sum_{k=1}^{Nm} \gamma_k^l \mathbf{E}^k \mathbf{Y}_k = \mathbf{0}, l = 0, \cdots, m-2$ and $\sum_{k=1}^{Nm} \gamma_k^{m-1} \mathbf{E}^k \mathbf{Y}_k = \mathbf{I}$, we then obtain for $Re(\phi) \geq 0$,

$$\begin{aligned}
\bar{\mathbf{Z}}(\phi) =& \mathbf{p} + \mathcal{V}_2 \mathbf{L}_0(0) \sum_{k=2}^{Nm} \mathbf{E}^k \mathbf{Y}_k + \mathcal{V}_2 \mathbf{L}_0(0) \sum_{k=2}^{Nm} \frac{\gamma_k \mathbf{E}^k \mathbf{Y}_k}{(\phi-\gamma_k)} \\
&+ \mathcal{V}_2 \sum_{l=1}^{m-1} \mathbf{L}_l(0) \left[\sum_{k=1}^{Nm} \gamma_k^l \mathbf{E}^k \mathbf{Y}_k + \sum_{k=1}^{Nm} \frac{\gamma_k^{l+1} \mathbf{E}^k \mathbf{Y}_k}{(\phi-\gamma_k)} \right] \\
=& \mathbf{p} + \mathcal{V}_2 \mathbf{L}_0(0) \sum_{k=2}^{Nm} \mathbf{E}^k \mathbf{Y}_k + \mathcal{V}_2 \mathbf{L}_0(0) \sum_{k=2}^{Nm} \frac{\gamma_k \mathbf{E}^k \mathbf{Y}_k}{(\phi-\gamma_k)} \\
&+ \mathcal{V}_2 \mathbf{C}_0 + \mathcal{V}_2 \sum_{l=1}^{m-1} \mathbf{L}_l(0) \sum_{k=2}^{Nm} \frac{\gamma_k^{l+1} \mathbf{E}^k \mathbf{Y}_k}{(\phi-\gamma_k)}.
\end{aligned} \tag{6.76}$$

In a similar way as on page 195 in Appendix A.10, we can check that for $j = 1, 2, \cdots, \bar{K}m$, $\lim_{\phi \to \gamma_j} (\phi - \gamma_j) \bar{\mathbf{Z}}(\phi) = 0$, or $\bar{\mathbf{Z}}(\phi)$ is analytic in the right half-plane $Re(\phi) \geq 0$.

The equation (6.75) shows us that we have a closed-form Laplace-Stieltjes transform for the steady-state buffer content at transition epochs. By inverting the j-th element of (6.75), we obtain for $x \geq 0$, $j = 1, 2, \cdots, N$,

$$\begin{aligned}
&P(W \leq x, X = j) \\
=& p_j + \sum_{k=\bar{K}m+1}^{Nm} \left[\mathcal{V}_2 \mathbf{L}_0(0) \mathbf{E}^k \mathbf{Y}_k \right]_j e^{\gamma_k x} - \sum_{k=\bar{K}m+1}^{Nm} \left[\mathcal{V}_2 \sum_{l=1}^{m-2} \mathbf{L}_l(0) \gamma_k^l \mathbf{E}^k \mathbf{Y}_k \right]_j \\
&+ \sum_{k=\bar{K}m+1}^{Nm} \left[\mathcal{V}_2 \sum_{l=1}^{m-1} \mathbf{L}_l(0) \gamma_k^l \mathbf{E}^k \mathbf{Y}_k \right]_j e^{\gamma_k x},
\end{aligned} \tag{6.77}$$

which shows us that the steady-state distribution of the buffer content at transition epochs is a mixture of exponentials and a concentration at 0. This structure is the same as the structure of the corresponding distribution in chapter 5.

6.5.2 The steady-state distribution of the buffer content in continuous time

The process $\{(V_t, J_t), t \geq 0\}$ is regenerative, where the regeneration points are the epochs at which the process leaves a state $(0, i)$ for some $i \in \mathcal{N}$. Since the times between regeneration points are non-arithmetic and have finite expectation as can be inferred from Theorem 6.5.1 and the finite means of inter-jump times of $\{J_t\}$, the process $\{(V_t, J_t), t \geq 0\}$ converges weakly to a random vector (V, J). Let

$$Z_i^*(\phi) = \lim_{t \to \infty} E\left(e^{-\phi V_t} \mathbf{1}(J_t = i)\right),$$

which is independent of initial conditions and will be determined using Abel's limit theorem.

Lemma 6.5.1

$$\frac{d}{d\eta}\gamma_1(1,\eta)\Big|_{\eta=0} = -\frac{\sum_{j=1}^{N} p_j \sum_{k=1}^{m} \frac{a_{jk}}{\mu_{jk}}}{\sum_{j=1}^{N} p_j r_j \sum_{k=1}^{m} \frac{a_{jk}}{\eta_{jk}}} = -\frac{1}{\sum_{j=1}^{N} \pi_j r_j}.$$

Proof. By the same argument as in the proof of Lemma 6.3.2 it is seen that there is a vector $\mathbf{v}(\eta)$ with $\mathbf{v}(0) = \mathbf{1}$ such that for $i = 1, 2, \cdots, N$,

$$\sum_{j=1}^{N} (\delta_{ij} - G_{ij}(\gamma_1(1,\eta),\eta)) v_j(\eta) = 0.$$

It is readily verified from this equation using the same procedure as in Lemma 6.3.2 that

$$\frac{d}{d\eta}\gamma_1(1,\eta)\Big|_{\eta=0} = -\frac{\sum_{i=1}^{N} p_i \sum_{j=1}^{N} \frac{d}{d\eta} G_{ij}(0,\eta)\Big|_{\eta=0}}{\sum_{i=1}^{N} p_i \sum_{j=1}^{N} \frac{d}{d\phi} G_{ij}(\phi,0)\Big|_{\phi=0}}$$

and the result follows from the definition of $\mathbf{G}$. ■

Let $\mathcal{V}_3 = -\sum_{j=1}^{N} \pi_j r_j$. Similar to the derivation of (6.70) we have

$$\lim_{\eta \downarrow 0} \eta Z_{il}(1,\phi,\eta,v) = -\mathcal{V}_3 \mathbf{V}(\phi)_l, \tag{6.78}$$

where the $N-$dimensional row vector $\mathbf{V}(\phi)$ is defined by

$$\mathbf{V}(\phi) = \phi \mathbf{S}_1^{-1} \sum_{l=0}^{m-1} \mathbf{L}_l(0) \sum_{k=1}^{Nm} \frac{\gamma_k^l \mathbf{E}^k \mathbf{Y}_k}{(\phi - \gamma_k)}.$$

The expression (6.78) shows that $\lim_{\eta \downarrow 0} \eta Z_{il}(\phi,\eta,v)$ does not depend on i and v.

By applying Abel's limit theorem to (6.62) and (6.63), relation (6.78) yields for $Re(\phi) \geq 0, j \in R^-$,

$$\begin{aligned} Z_j^*(\phi) =& \lim_{\eta \downarrow 0} \eta Z_{ij}^*(\phi, \eta, v) \\ =& \frac{1}{r_j} \sum_{l=1}^{N} \lim_{\eta \downarrow 0} \eta Z_{il}(1, \phi, \eta, v) P_{lj} \sum_{k=1}^{m} \frac{a_{jk}}{(\phi + \alpha_{jk}(\eta))} \\ &+ \frac{1}{r_j} \sum_{l=1}^{N} \sum_{k=1}^{m} \lim_{\eta \downarrow 0} \eta Z_{il}(1, -\alpha_{jk}(\eta), \eta, v) P_{lj} \frac{\phi}{(\phi + \alpha_{jk}(\eta))} \frac{a_{jk}}{\alpha_{jk}(\eta)} \\ =& -\frac{\mathcal{V}_3}{r_j} \sum_{l=1}^{N} \mathbf{V}(\phi)_l P_{lj} \sum_{k=1}^{m} \frac{a_{jk}}{(\phi + \alpha_{jk})} \\ &- \frac{\mathcal{V}_3}{r_j} \sum_{l=1}^{N} \sum_{k=1}^{m} \mathbf{V}(-\alpha_{jk})_l P_{lj} \frac{\phi}{(\phi + \alpha_{jk})} \frac{a_{jk}}{\alpha_{jk}}, \end{aligned} \tag{6.79}$$

and for $j \in R^+$,

$$\begin{aligned} Z_j^*(\phi) =& \lim_{\eta \downarrow 0} \eta Z_{ij}^*(\phi, \eta, v) \\ =& \frac{1}{r_j} \sum_{l=1}^{N} \lim_{\eta \downarrow 0} \eta Z_{il}(1, \phi, \eta, v) P_{lj} \sum_{k=1}^{m} \frac{a_{jk}}{\phi + \alpha_{jk}(\eta)} \\ =& -\frac{\mathcal{V}_3}{r_j} \sum_{l=1}^{N} V(\phi)_l P_{lj} \sum_{k=1}^{m} \frac{a_{jk}}{\phi + \alpha_{jk}}. \end{aligned} \tag{6.80}$$

Noting that

$$\begin{aligned} V(\phi)_l =& \left[\mathbf{S}_1^{-1} \mathbf{L}_0(0) \sum_{k=1}^{Nm} \frac{\phi \mathbf{E}^k \mathbf{Y}_k}{(\phi - \gamma_k)} \right]_l + \left[\mathbf{S}_1^{-1} \mathbf{C}_0 \right]_l + \left[\sum_{l=1}^{m-1} \mathbf{S}_1^{-1} \mathbf{L}_l(0) \sum_{k=1}^{Nm} \frac{\gamma_k^{l+1} \mathbf{E}^k \mathbf{Y}_k}{(\phi - \gamma_k)} \right]_l \\ =& \mathbf{Y}_{il} + \left[\mathbf{S}_1^{-1} \mathbf{L}_0(0) \sum_{k=2}^{Nm} \frac{\phi \mathbf{E}^k \mathbf{Y}_k}{(\phi - \gamma_k)} \right]_l + \left[\mathbf{S}_1^{-1} \mathbf{C}_0 \right]_l \\ &+ \left[\sum_{l=1}^{m-1} \mathbf{S}_1^{-1} \mathbf{L}_l(0) \sum_{k=1}^{Nm} \frac{\gamma_k^{l+1} \mathbf{E}^k \mathbf{Y}_k}{(\phi - \gamma_k)} \right]_l \end{aligned}$$

where $\mathbf{Y}_{1l} = \frac{p_l}{\mathcal{V}_1}$ and

$$-\frac{\mathcal{V}_3}{r_j \mathcal{V}_1} \sum_{l=1}^{N} p_l P_{lj} \sum_{k=1}^{m} \frac{a_{jk}}{\phi + \alpha_{jk}} = -\frac{\mathcal{V}_3}{r_j \mathcal{V}_1} p_j \sum_{k=1}^{m} \frac{a_{jk}}{\phi + \alpha_{jk}},$$

we then obtain for $Re(\phi) \geq 0$, $j \in R^-$,

$$
\begin{aligned}
&Z_j^*(\phi) \\
&= -\frac{\mathcal{V}_3 p_j}{r_j \mathcal{V}_1} \sum_{k=1}^{m} \frac{a_{jk}}{\phi + \alpha_{jk}} - \frac{\mathcal{V}_3}{r_j} \sum_{l=1}^{N} \left[\mathbf{S}_1^{-1}\mathbf{C}_0\right]_l P_{lj} \sum_{k=1}^{m} \frac{a_{jk}}{\phi + \alpha_{jk}} \\
&\quad - \frac{\mathcal{V}_3}{r_j} \sum_{l=1}^{N} \left[\mathbf{S}_1^{-1}\mathbf{L}_0(0) \sum_{k_1=2}^{Nm} \frac{\phi \mathbf{E}^{k_1}\mathbf{Y}_{k_1}}{(\phi - \gamma_{k_1})}\right]_l P_{lj} \sum_{k_2=1}^{m} \frac{a_{jk_2}}{(\phi + \alpha_{jk_2})} \\
&\quad - \frac{\mathcal{V}_3}{r_j} \sum_{l=1}^{N} \left[\sum_{l_1=1}^{m-1} \mathbf{S}_1^{-1}\mathbf{L}_l(0) \sum_{k_1=2}^{Nm} \frac{\gamma_{k_1}^{l_1+1}\mathbf{E}^{k_1}\mathbf{Y}_{k_1}}{(\phi - \gamma_{k_1})}\right]_l P_{lj} \sum_{k_2=1}^{m} \frac{a_{jk_2}}{(\phi + \alpha_{jk_2})} \\
&\quad - \frac{\mathcal{V}_3}{r_j} \sum_{l=1}^{N} \sum_{k=2}^{m} \mathbf{V}(-\alpha_{jk})_l P_{lj} \frac{\phi}{(\phi + \alpha_{jk})} \frac{a_{jk}}{\alpha_{jk}},
\end{aligned}
\tag{6.81}
$$

and for $Re(\phi) \geq 0$, $j \in R^+$,

$$
\begin{aligned}
&Z_j^*(\phi) \\
&= -\frac{\mathcal{V}_3 p_j}{r_j \mathcal{V}_1} \sum_{k=1}^{m} \frac{a_{jk}}{\phi + \alpha_{jk}} - \frac{\mathcal{V}_3}{r_j} \sum_{l=1}^{N} \left[\mathbf{S}_1^{-1}\mathbf{C}_0\right]_l P_{lj} \sum_{k=1}^{m} \frac{a_{jk}}{\phi + \alpha_{jk}} \\
&\quad - \frac{\mathcal{V}_3}{r_j} \sum_{l=1}^{N} \left[\mathbf{S}_1^{-1}\mathbf{L}_0(0) \sum_{k_1=2}^{Nm} \frac{\phi \mathbf{E}^{k_1}\mathbf{Y}_{k_1}}{(\phi - \gamma_{k_1})}\right]_l P_{lj} \sum_{k_2=1}^{m} \frac{a_{jk_2}}{(\phi + \alpha_{jk_2})} \\
&\quad - \frac{\mathcal{V}_3}{r_j} \sum_{l=1}^{N} \left[\sum_{l_1=1}^{m-1} \mathbf{S}_1^{-1}\mathbf{L}_l(0) \sum_{k_1=2}^{Nm} \frac{\gamma_{k_1}^{l_1+1}\mathbf{E}^{k_1}\mathbf{Y}_{k_1}}{(\phi - \gamma_{k_1})}\right]_l P_{lj} \sum_{k_2=1}^{m} \frac{a_{jk_2}}{(\phi + \alpha_{jk_2})}.
\end{aligned}
\tag{6.82}
$$

Like in subsection 6.5.1, we get a closed form Laplace-Stieltjes transform of the steady-state buffer content in continuous time. By inverting (6.81) and (6.82) we obtain for $x \geq 0$, $j = 1, 2, \cdots, N$,

$$
\begin{aligned}
&P(V \leq x, J = j) \\
&= -\frac{\mathcal{V}_3 p_j}{r_j \mathcal{V}_1} \sum_{k=1}^{m} \frac{a_{jk}}{\alpha_{jk}} - \frac{\mathcal{V}_3}{r_j} \sum_{l=1}^{N} \left[\mathbf{S}_1^{-1}\mathbf{C}_0\right]_l P_{lj} \sum_{k=1}^{m} \frac{a_{jk}}{\alpha_{jk}} \\
&\quad - \frac{\mathcal{V}_3}{r_j} \sum_{k_1=\bar{K}m+1}^{Nm} \sum_{l=1}^{N} \left[\mathbf{S}_1^{-1}\mathbf{L}_0(0)\mathbf{E}^{k_1}\mathbf{Y}_{k_1}\right]_l P_{lj} \sum_{k_2=1}^{m} \frac{a_{jk_2}}{(\gamma_{k_1} + \alpha_{jk_2})} e^{\gamma_{k_1} x} \\
&\quad + \frac{\mathcal{V}_3}{r_j} \sum_{l=1}^{N} \left[\sum_{l_1=1}^{m-1} \mathbf{S}_1^{-1}\mathbf{L}_l(0) \sum_{k_1=2}^{Nm} \gamma_{k_1}^{l_1}\mathbf{E}^{k_1}\mathbf{Y}_{k_1}\right]_l P_{lj} \sum_{k_2=1}^{m} \frac{a_{jk_2}}{\alpha_{jk_2}} \\
&\quad - \frac{\mathcal{V}_3}{r_j} \sum_{k_1=\bar{K}m+1}^{Nm} \sum_{l=1}^{N} \left[\sum_{l_1=1}^{m-1} \mathbf{S}_1^{-1}\mathbf{L}_l(0)\gamma_{k_1}^{l_1}\mathbf{E}^{k_1}\mathbf{Y}_{k_1}\right]_l P_{lj} \sum_{k_2=1}^{m} \frac{a_{jk_2}}{(\gamma_{k_1} + \alpha_{jk_2})} e^{\gamma_{k_1} x}.
\end{aligned}
\tag{6.83}
$$

Since $\mathcal{V}_1 = -\left(\sum_{j=1}^{N}\pi_j r_j\right)\left(\sum_{i=1}^{m} p_i \sum_{k=1}^{m}\frac{a_{ik}}{\mu_{ik}}\right)$ and $\pi_j = \frac{p_j \sum_{k=1}^{m}\frac{a_{jk}}{\mu_{jk}}}{\sum_{i=1}^{N} p_i \sum_{k=1}^{m}\frac{a_{ik}}{\mu_{ik}}}$, then

$$\frac{\mathcal{V}_3 p_j}{r_j \mathcal{V}_1}\sum_{k=1}^{m}\frac{a_{jk}}{\alpha_{jk}} = \pi_j.$$

It should also be noted that, since $\gamma_1 = 0$,

$$\begin{aligned}
&\frac{\mathcal{V}_3}{r_j}\sum_{l=1}^{N}\left[\sum_{l_1=1}^{m-1}\mathbf{S}_1^{-1}\mathbf{L}_l(0)\sum_{k_1=2}^{Nm}\gamma_{k_1}^{l_1}\mathbf{E}^{k_1}\mathbf{Y}_{k_1}\right]_l P_{lj}\sum_{k_2=1}^{m}\frac{a_{jk_2}}{\alpha_{jk_2}}\\
=&\frac{\mathcal{V}_3}{r_j}\sum_{l=1}^{N}\left[\sum_{l_1=1}^{m-1}\mathbf{S}_1^{-1}\mathbf{L}_l(0)\sum_{k_1=1}^{Nm}\gamma_{k_1}^{l_1}\mathbf{E}^{k_1}\mathbf{Y}_{k_1}\right]_l P_{lj}\sum_{k_2=1}^{m}\frac{a_{jk_2}}{\alpha_{jk_2}}\\
=&\frac{\mathcal{V}_3}{r_j}\sum_{l=1}^{N}\mathbf{S}_1^{-1}\mathbf{C}_0 P_{lj}\sum_{k_2=1}^{m}\frac{a_{jk_2}}{\alpha_{jk_2}},
\end{aligned}$$

so that the second and the fifth term of the right-hand side of (6.83) are cancelled.

Equation (6.83) can thus be written as follows. For $x \geq 0$, $j = 1, 2, \cdots, N$,

$$\begin{aligned}
&P(V \leq x, J = j)\\
=&\pi_j - \frac{\mathcal{V}_3}{r_j}\sum_{k_1=\bar{K}m+1}^{Nm}\sum_{l=1}^{N}\left[\mathbf{S}_1^{-1}\mathbf{L}_0(0)\mathbf{E}^{k_1}\mathbf{Y}_{k_1}\right]_l P_{lj}\sum_{k_2=1}^{m}\frac{a_{jk_2}}{(\gamma_{k_1}+\alpha_{jk_2})}e^{\gamma_{k_1}x}\\
&-\frac{\mathcal{V}_3}{r_j}\sum_{k_1=\bar{K}m+1}^{Nm}\sum_{l=1}^{N}\left[\sum_{l_1=1}^{m-1}\mathbf{S}_1^{-1}\mathbf{L}_l(0)\gamma_{k_1}^{l_1}\mathbf{E}^{k_1}\mathbf{Y}_{k_1}\right]_l P_{lj}\sum_{k_2=1}^{m}\frac{a_{jk_2}}{(\gamma_{k_1}+\alpha_{jk_2})}e^{\gamma_{k1}x},
\end{aligned} \tag{6.84}$$

which shows us that the steady-state distribution of the buffer content in continuous time is a mixture of exponentials and a concentration at 0. This structure is the same as the structure of such distribution of the Markovian fluid flow model studied in chapter 5, and aggrees the structure of such distribution of the semi-Markovian fluid flow model in [8].

6.6 Numerical examples

In this section we give some examples of the probability distribution of the buffer content in continuous time for two semi-Markovian fluid flow models, where:

- $N = 5$,

- the transition matrix of the Markov chain $\{X_n, n \geq 1\}$ is given by

$$\mathbf{P} = \begin{pmatrix} 0 & 0.25 & 0.25 & 0.25 & 0.25 \\ 0.2 & 0 & 0.4 & 0 & 0.4 \\ 0.333 & 0.333 & 0 & 0.111 & 0.222 \\ 0.125 & 0.25 & 0.125 & 0 & 0.5 \\ 0.5 & 0.333 & 0.1667 & 0 & 0 \end{pmatrix}, \tag{6.85}$$

- for $j \in \mathcal{N},\ k \in \mathcal{M},$ the parameters μ_{jk} are

$$\begin{aligned} \mu_{11} &= 5, \quad \mu_{12} = 3, \\ \mu_{21} &= 4, \quad \mu_{22} = 6, \\ \mu_{31} &= 2, \quad \mu_{32} = 5/2, \\ \mu_{41} &= 3/2, \quad \mu_{42} = 1, \\ \mu_{51} &= 7/2, \quad \mu_{52} = 9/2. \end{aligned} \tag{6.86}$$

In subsection 6.6.1 we give the numerical results for the model where for $j = 1, \cdots, N$, H_j is hyper-exponentially distributed, and in subsection 6.6.2 we consider the model where H_j is hypo-exponentially distributed.

The steady-state distribution function $P\{V \leq x, J = j\}$ of the buffer content in continuous time is given explicitly by (6.84). Figures 6.1 and 6.8 are obtained from (6.84), and in subsections 6.6.1 and 6.6.2 we show that all the time-dependent distributions converge to the steady-state distribution. The time-dependent distribution function of the buffer content

$$F_{ij}(x, t, v) = P(V_t \leq x, J_t = j | X_1 = i, V_0 = v)$$

can be obtained by applying the numerical inversion algorithm written in [3] to (6.66) - (6.69). Figures 6.2 - 6.7 and figures 6.9 - 6.13 give some results on the numerical inversion for the models with the transition matrix $\mathbf{P}$ given by (6.85) with various values of the initial buffer content, the traffic intensity, and the net input rates.

6.6.1 Hyper-exponential case

In the first model we assume that for $j \in \mathcal{N}$, H_j, the time the process $\{J_t,\ t \geq 0\}$ spends in state j before making a transition to a different state is hyper-exponentially distributed with parameters μ_{jk} for $j \in \mathcal{N}, k \in \mathcal{M}$ which is given by (6.86) and the weight parameters are

$$\begin{aligned} a_{11} &= 3/4, \quad a_{12} = 1/4, \\ a_{21} &= 3/5, \quad a_{22} = 2/5, \\ a_{31} &= 5/6, \quad a_{32} = 1/6, \\ a_{41} &= 1/2, \quad a_{42} = 1/2, \\ a_{51} &= 2/3, \quad a_{52} = 1/3. \end{aligned}$$

It follows that the stationary probability distribution of $\{J_t,\, t \geq 0\}$ is given by

$$\pi_1 = 0.16844,$$
$$\pi_2 = 0.14716,$$
$$\pi_3 = 0.29064,$$
$$\pi_4 = 0.20606,$$
$$\pi_5 = 0.18768.$$

The net input rates are given by

$$r_1 = -2, r_2 = -1.25, r_3 = -4, r_4 = 1.5, r_5 = 3.5.$$

The graphs of the steady-state distribution functions of the buffer content in continuous time $P\{V \leq x, J = j\}$ for $j = 1, \cdots, 5$, can be seen in figure 6.1. The function values for some values of x, are given in table 6.1. The figure and the table show that

$$\lim_{x\to\infty} P\{V \leq x, J = i\} = \pi_i, \quad i = 1, 2, \cdots, N.$$

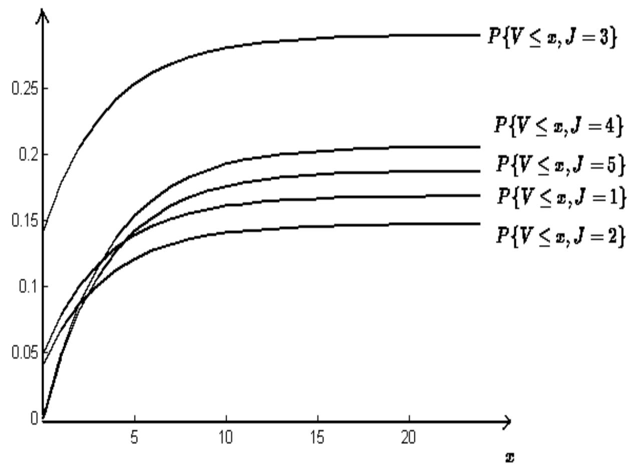

Figure 6.1: The steady-state distribution functions of the buffer content in continuous time of the model with hyper-exponential H_j, $j = 1, \cdots, 5$.

x	$P\{V \leq x, J = j\}$				
	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
0	0.0552	0.0464	0.1529	0	0
0.5	0.0730	0.0623	0.1744	0.0321	0.0294
1	0.0879	0.0756	0.1925	0.0589	0.0542
1.5	0.1004	0.0867	0.2077	0.0815	0.0750
2	0.1110	0.0961	0.2205	0.1006	0.0925
2.5	0.1199	0.1040	0.2313	0.1167	0.1073
3	0.1274	0.1106	0.2404	0.1303	0.1197
3.5	0.1337	0.1163	0.2482	0.1419	0.1302
⋮	⋮	⋮	⋮	⋮	⋮
10	0.1644	0.1436	0.2857	0.1986	0.1811
10.5	0.1650	0.1441	0.2865	0.1998	0.1821
11	0.1656	0.1446	0.2871	0.2007	0.1829
11.5	0.1660	0.1450	0.2877	0.2015	0.1836
⋮	⋮	⋮	⋮	⋮	⋮
18	0.16838	.14710	0.29056	0.20595	0.18757
18.5	0.16839	0.14711	0.29057	0.20597	0.18759
19	0.16839	0.14712	0.29058	0.20598	0.18760
19.5	0.16840	0.14713	0.29059	0.20599	0.18761
20	0.16841	0.14713	0.29060	0.20601	0.18762
stationary probability π_j	0.16844	0.14716	0.29064	0.20606	0.18768

Table 6.1: The steady-state distribution functions values for the hyper-exponential case

Figures 6.2 - 6.4 show the behavior of $F_{44}(x,t,v)$ for a fixed value of ρ but for different values of v. These graphs show that, for the same net input rates and for the same value of traffic intensity, the time-dependent distribution functions convergence to the steady-state distribution function faster as v is closer to π_4. Figures 6.5 - 6.7 show the behavior of $F_{23}(x,t,v)$ for the same v and the same net input rates, but for different values of the traffic intensity ρ.

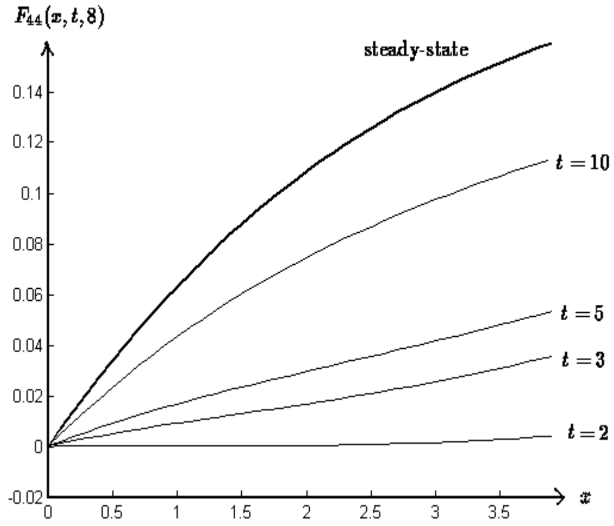

Figure 6.2: The time-dependent distribution function $F_{44}(x,t,8) = P(V_t \leq x, J_t = 4|X_1 = 4, V_0 = 8)$ for different values of t, for the model with $\rho = 0.8804$, and the net input rates: $r_1 = -2, r_2 = -1.25, r_3 = -4, r_4 = 1.5, r_5 = 3.5$.

x	$F_{44}(x,2,8)$	$F_{44}(x,3,8)$	$F_{44}(x,5,8)$	$F_{44}(x,10,8)$	$P\{V \leq x, J=4\}$
0	0	0	0	0	1.4e-05
0.5	0.00232	0.00799	0.01585	0.02269	0.02608
1	0.00418	0.01452	0.02944	0.042451	0.04887
1.5	0.00645	0.02038	0.04135	0.05974	0.06878
2	0.00947	0.02619	0.05203	0.07492	0.08619
2.5	0.01339	0.03238	0.06185	0.08830	0.10141
3	0.00148	0.00970	0.03482	0.07742	0.11471
3.5	0.00258	0.01283	0.04039	0.08608	0.12633

Table 6.2: The values of $F_{44}(x,t,8)$ and the steady-state distribution function in figure 6.2 on some values of x

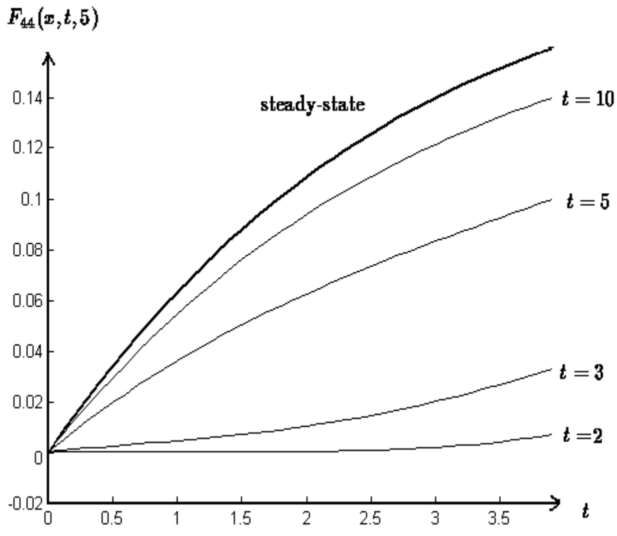

Figure 6.3: The time-dependent distribution function $F_{44}(x,t,5) = P(V_t \leq x, J_t = 4 | X_1 = 4, V_0 = 5)$ for different values of t, for the model with $\rho = 0.8804$.

x	$F_{44}(x, 2, 8)$	$F_{44}(x, 3, 8)$	$F_{44}(x, 5, 8)$	$F_{44}(x, 10, 8)$	$P\{V \leq x, J = 4\}$
0	0	0	0	0	1.4e-05
0.5	5.3e-06	0.00132	0.00739	0.01741	0.02608
1	3.8e-05	0.00238	0.01360	0.03251	0.04887
1.5	0.00013	0.00358	0.01914	0.04575	0.06878
2	0.00035	0.00514	0.02436	0.05746	0.08619
2.5	0.00077	0.00716	0.02952	0.06794	0.10141
3	0.00148	0.00970	0.03482	0.07742	0.11471
3.5	0.00258	0.01283	0.04039	0.08608	0.12633

Table 6.3: The values of $F_{44}(x, t, 5)$ and the steady-state distribution function in figure 6.3 on some values of x

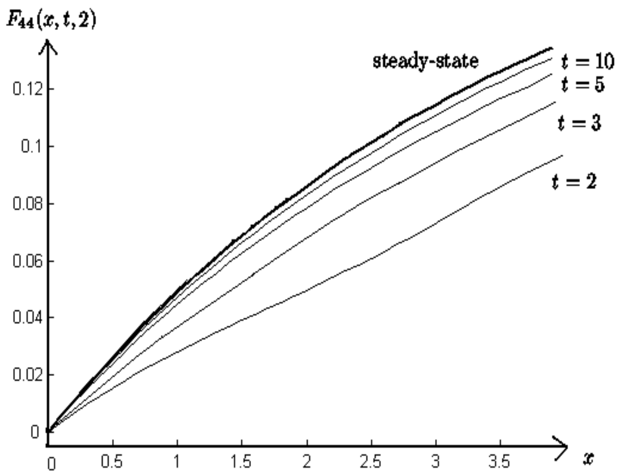

Figure 6.4: The time-dependent distribution function $F_{44}(x, t, 2) = P(V_t \leq x, J_t = 4 | X_1 = 4, V_0 = 2)$ for different values of t, for the model with $\rho = 0.8804$.

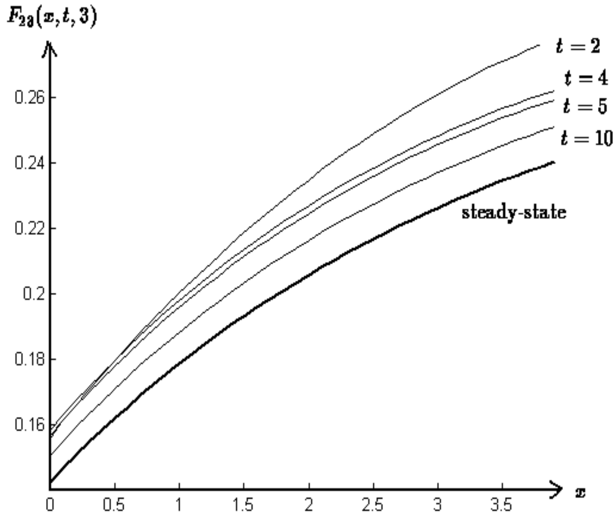

Figure 6.5: The time-dependent distribution function $F_{23}(x,t,3) = P(V_t \leq x, J_t = 3|X_1 = 2, V_0 = 3)$ for different values of t, for the model with $\rho = 0.82$, and the net input rates: $r_1 = -2, r_2 = -1.25, r_3 = -4, r_4 = 1.5, r_5 = 3.5$.

x	$F_{23}(x,2,3)$	$F_{23}(x,4,3)$	$F_{23}(x,5,3)$	$F_{23}(x,10,3)$	$P\{V \leq x, J = 3\}$
0	0.15585	0.15815	0.15662	0.15034	0.17745
0.5	0.17947	0.17956	0.17776	0.17075	0.19959
1	0.20032	0.19790	0.19587	0.18829	0.21693
1.5	0.21872	0.21362	0.21138	0.20338	0.23070
2	0.23493	0.22708	0.22466	0.21637	0.24176
2.5	0.24906	0.23857	0.23600	0.22755	0.25068
3	0.26115	0.24836	0.24566	0.23717	0.25793
3.5	0.27125	0.25667	0.25387	0.24543	0.26383

Table 6.4: The values of $F_{23}(x,t,3)$ and the steady-state distribution function in figure 6.5 on some values of x

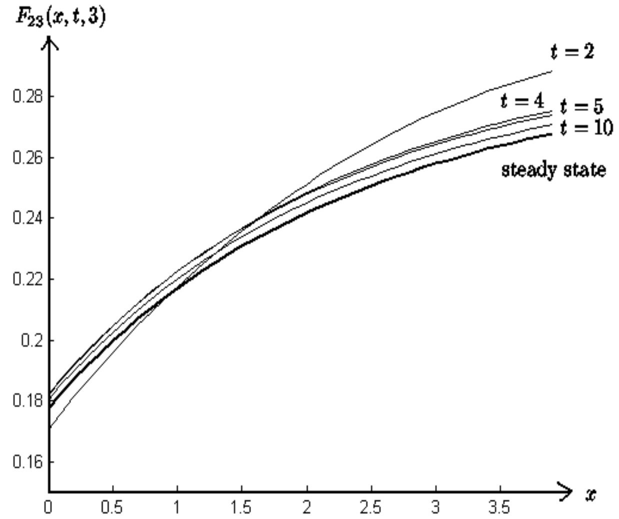

Figure 6.6: The time-dependent distribution function $F_{23}(x,t,3) = P(V_t \leq x, J_t = 3|X_1 = 2, V_0 = 3)$ for different values of t, for the model with $\rho = 0.835$, and the net input rates: $r_1 = -2, r_2 = -1.25, r_3 = -4, r_4 = 1.5, r_5 = 3.5$.

x	$F_{23}(x,2,3)$	$F_{23}(x,4,3)$	$F_{23}(x,5,3)$	$F_{23}(x,10,3)$	$P\{V \leq x, J = 3\}$
0	0.17030	0.18159	0.18195	0.18015	0.17745
0.5	0.19587	0.20446	0.20465	0.20255	0.19959
1	0.21738	0.22245	0.22242	0.22006	0.21693
1.5	0.23566	0.23680	0.23652	0.23393	0.23070
2	0.25122	0.24835	0.24780	0.24502	0.24176
2.5	0.26424	0.25769	0.25688	0.25393	0.25068
3	0.27484	0.26526	0.26419	0.26112	0.25793
3.5	0.28312	0.27138	0.27008	0.26693	0.26383

Table 6.5: The values of $F_{23}(x,t,3)$ and the steady-state distribution function in figure 6.6 on some values of x

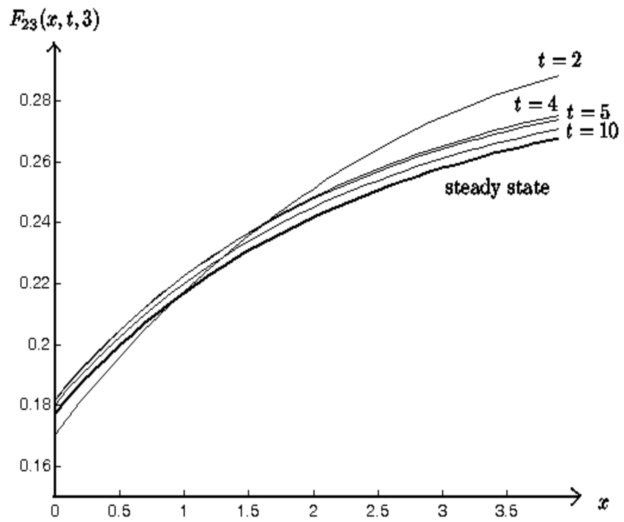

Figure 6.7: The time-dependent distribution function $F_{23}(x,t,3) = P(V_t \leq x, J_t = 3|X_1 = 2, V_0 = 3)$ for different values of t, for the model with $\rho = 0.87$, and the net input rates: $r_1 = -2, r_2 = -1.25, r_3 = -4, r_4 = 1.5, r_5 = 3.5$.

x	$F_{23}(x,2,3)$	$F_{23}(x,4,3)$	$F_{23}(x,5,3)$	$F_{23}(x,10,3)$	$P\{V \leq x, J = 3\}$
0	0.17030	0.18159	0.18195	0.18015	0.17745
0.5	0.19587	0.20446	0.20465	0.20255	0.19959
1	0.21738	0.22245	0.22242	0.22006	0.21693
1.5	0.23566	0.23680	0.23652	0.23393	0.23070
2	0.25122	0.24835	0.24780	0.24502	0.24176
2.5	0.26424	0.25769	0.25688	0.25393	0.25068
3	0.27484	0.26526	0.26419	0.26112	0.25793
3.5	0.28312	0.27138	0.27008	0.26693	0.26383

Table 6.6: The values of $F_{23}(x,t,3)$ and the steady-state distribution function in figure 6.7 on some values of x

6.6.2 Hypo-exponential case

In second model we assume that for $j \in \mathcal{N}$, H_j, the amount of time the process $\{J_t, t \geq 0\}$ spends in state j before making a transition to a different state is hypo-exponentially distributed with the parameters μ_{jk} for $j \in \mathcal{N}, k \in \mathcal{M}$ which is given by (6.86) and the parameters a_{jk} satisfy (6.2).

The stationary probability distribution of $\{J_t, t \geq 0\}$ is then given by

$$\pi_1 = 0.19428,$$
$$\pi_2 = 0.14281,$$
$$\pi_3 = 0.27309,$$
$$\pi_4 = 0.20797,$$
$$\pi_5 = 0.18183.$$

The graphs of the steady-state distribution functions of the buffer content in continuous time $P\{V \leq x, J = j\}$ for $j = 1, \cdots, 5$, can be seen in figure 6.8. The function values for some values of x, are given in table 6.7. The figure and the table show that

$$\lim_{x \to \infty} P\{V \leq x, J = i\} = \pi_i, \quad i = 1, 2, \cdots, N.$$

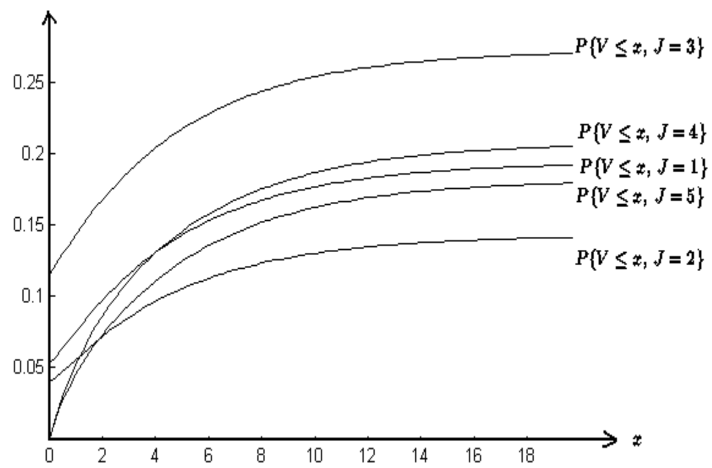

Figure 6.8: The steady-state distribution functions of the buffer content in continuous time of the model with hypo-exponential H_j, $j = 1, \cdots, 5$, the net input rates: $r_1 = -2, r_2 = -1.25, r_3 = -4, r_4 = 1.5, r_5 = 3.5$.

Figures 6.9 and 6.10 show the behavior of $F_{52}(x, t, v)$ for the same net input rates and for a fixed value of ρ but for different values of v. As for the hyper-exponential case, we see that for the same net input rates and for the same value of the traffic intensity, the time-dependent distribution functions convergence to the steady-state distribution function faster as v is closer to the stationary probability.

Moreover, figures 6.11 - 6.13 show the behavior of $F_{52}(x, t, v)$ for the same traffic intensity $\rho = 0.822$ and for the same initial buffer content v, but for different net input rates.

x	$P\{V \leq x, J = j\}$				
	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
0	0.0650	0.0463	0.1456	0	0
0.5	0.0775	0.0550	0.1597	0.0211	0.0188
1	0.0903	0.0644	0.1729	0.0422	0.0371
1.5	0.1023	0.0734	0.1849	0.0620	0.0540
2	0.1132	0.0816	0.1955	0.0799	0.0691
2.5	0.1229	0.0889	0.2049	0.0958	0.0825
3	0.1315	0.0954	0.2132	0.1098	0.0944
3.5	0.1392	0.1012	0.2206	0.1220	0.1049
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	0.1828	0.1352	0.2636	0.1925	0.1678
10.5	0.1842	0.1362	0.2647	0.1944	0.1695
11	0.1855	0.1370	0.2657	0.1961	0.1710
11.5	0.1865	0.1377	0.2666	0.1975	0.1723
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
18	0.19391	0.14252	0.27274	0.20739	0.18131
18.5	0.19395	0.14256	0.27278	0.20746	0.18137
19	0.19399	0.14259	0.27282	0.20753	0.18143
19.5	0.19403	0.14262	0.27285	0.20758	0.18148
20	0.19406	0.14264	0.27288	0.20763	0.18152
stationary probability π_j	0.19428	0.14281	0.27309	0.20797	0.18183

Table 6.7: The steady-state distribution functions values for the hypo-exponential case

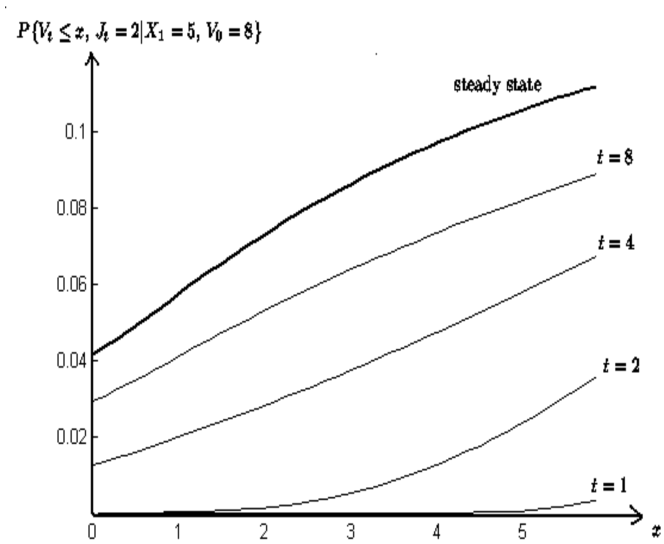

Figure 6.9: The time-dependent distribution function $F_{52}(x,t,8)$ for different values of t, $\rho = 0.8577$, the net input rates: $r_1 = -2, r_2 = -1.25, r_3 = -4, r_4 = 1.5, r_5 = 3.5$.

x	$F_{52}(x,1,5)$	$F_{52}(x,2,5)$	$F_{52}(x,3,5)$	$F_{52}(x,6,5)$	$P\{V \le x, J = 2\}$
0	0	9.7e-08	0.00368	0.02353	0.04151
0.6	0	2.5e-05	0.00614	0.02938	0.05069
1	0	0.00013	0.00822	0.03368	0.05737
1.6	0	0.00061	0.01196	0.04013	0.06708
2	0	0.00131	0.01488	0.04432	0.07311
2.6	0	0.00318	0.01986	0.05040	0.08138
3	0	0.00510	0.02355	0.05433	0.08637
3.6	5.0e-09	0.00912	0.02962	0.06008	0.09314
4	6.0e-08	0.01256	0.03398	0.06383	0.09721
4.6	0.00010	0.01877	0.04093	0.06936	0.10271
5	0.00051	0.02365	0.04580	0.07299	0.10601
5.6	0.00213	0.03208	0.05337	0.07838	0.11046

Table 6.8: The values of $F_{52}(x,t,4)$ and the steady-state distribution function in figure 6.9 on some values of x

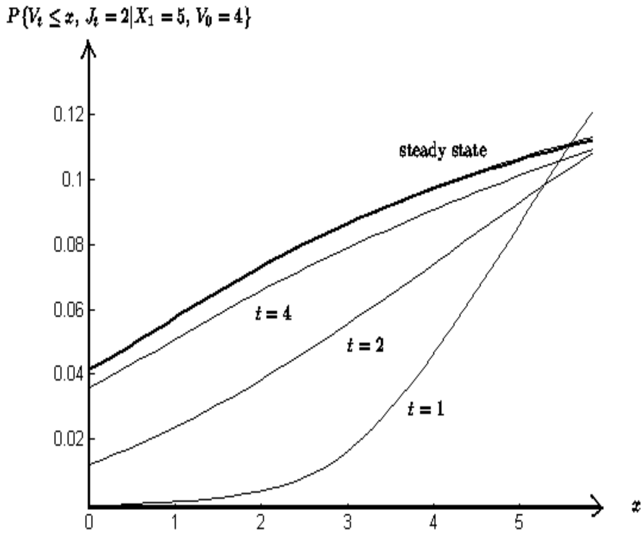

Figure 6.10: The time-dependent distribution function $F_{52}(x, t, 4)$ for different values of t, $\rho = 0.8577$, the net input rates: $r_1 = -2, r_2 = -1.25, r_3 = -4, r_4 = 1.5, r_5 = 3.5$.

x	$F_{52}(x, 1, 5)$	$F_{52}(x, 2, 5)$	$F_{52}(x, 3, 5)$	$F_{52}(x, 6, 5)$	$P\{V \le x, J = 2\}$
0	6.0e-8	0.01213	0.02945	0.04001	0.04151
0.6	0.00010	0.01851	0.03752	0.04902	0.05069
1	0.00051	0.02352	0.04335	0.05559	0.05737
1.6	0.00213	0.03205	0.05207	0.06518	0.06708
2	0.00408	0.03831	0.05776	0.07117	0.07311
2.6	0.00926	0.04838	0.06615	0.07945	0.08138
3	0.01606	0.05545	0.07166	0.08451	0.08637
3.6	0.03243	0.06646	0.07983	0.09145	0.09314
4	0.04628	0.07398	0.08519	0.09568	0.09721
4.6	0.06978	0.08537	0.09310	0.10149	0.10271
5	0.09478	0.09663	0.09824	0.10504	0.10601
5.6	0.11123	0.10391	0.10571	0.10994	0.11046

Table 6.9: The values of $F_{52}(x, t, 4)$ and the steady-state distribution function in figure 6.10 on some values of x

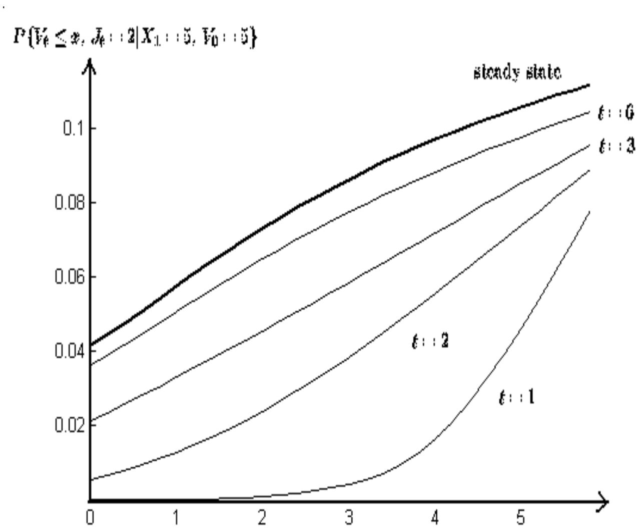

Figure 6.11: The time-dependent distribution function $F_{52}(x, t, 5)$ for different values of t, $\rho = 0.822$, the net input rates: $r_1 = -2, r_2 = -1.25, r_3 = -4, r_4 = 1.5, r_5 = 3.5$.

x	$F_{52}(x, 1, 5)$	$F_{52}(x, 2, 5)$	$F_{52}(x, 3, 5)$	$F_{52}(x, 6, 5)$	$P\{V \leq x, J = 2\}$
0	0	0.00501	0.02121	0.03612	0.04151
0.6	0	0.00908	0.02791	0.04440	0.05069
1	6.0e-008	0.01254	0.03282	0.05045	0.05737
1.6	0.00010	0.01876	0.04037	0.05933	0.06708
2	0.00051	0.02365	0.04548	0.06492	0.07311
2.6	0.00213	0.03208	0.05325	0.07272	0.08138
3	0.00408	0.03832	0.05850	0.07754	0.08637
3.6	0.00926	0.04838	0.06649	0.08425	0.09314
4	0.01606	0.05545	0.07185	0.08841	0.09721
4.6	0.03243	0.06646	0.07989	0.09422	0.10271
5	0.04628	0.07398	0.08544	0.09784	0.10601
5.6	0.06979	0.08537	0.09341	0.10293	0.11046

Table 6.10: The values of $F_{52}(x, t, 5)$ and the steady-state distribution function in figure 6.11 on some values of x

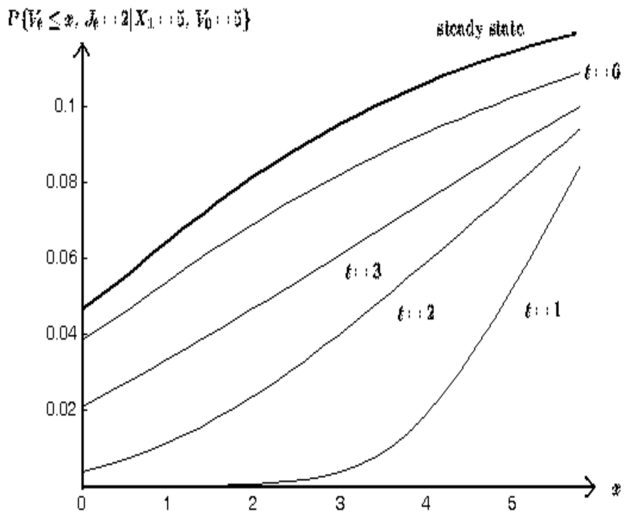

Figure 6.12: The time-dependent distribution function $F_{52}(x, t, 5)$ for different values of t, $\rho = 0.822$, the net input rates: $r_1 = -1.61, r_2 = -1.36, r_3 = -4.11, r_4 = 1.38, r_5 = 3.38$.

x	$F_{52}(x,1,5)$	$F_{52}(x,2,5)$	$F_{52}(x,3,5)$	$F_{52}(x,6,5)$	$P\{V \leq x, J = 2\}$
0	0	0.00571	0.02166	0.03657	0.04296
0.6	0	0.00988	0.02837	0.04517	0.05272
1	9.0e-007	0.01336	0.03325	0.05138	0.05974
1.6	0.00018	0.01950	0.04074	0.06041	0.06984
2	0.00069	0.02409	0.04580	0.06605	0.07605
2.6	0.00249	0.03208	0.05350	0.07387	0.08449
3	0.00452	0.03816	0.05871	0.07868	0.08955
3.6	0.00884	0.04809	0.06665	0.08535	0.09636
4	0.01538	0.05514	0.07200	0.08948	0.10042
4.6	0.03247	0.06624	0.08007	0.09523	0.10586
5	0.04711	0.07390	0.08544	0.09882	0.10910
5.6	0.07195	0.08562	0.09341	0.10387	0.11345

Table 6.11: The values of $F_{52}(x, t, 5)$ and the steady-state distribution function in figure 6.12 on some values of x

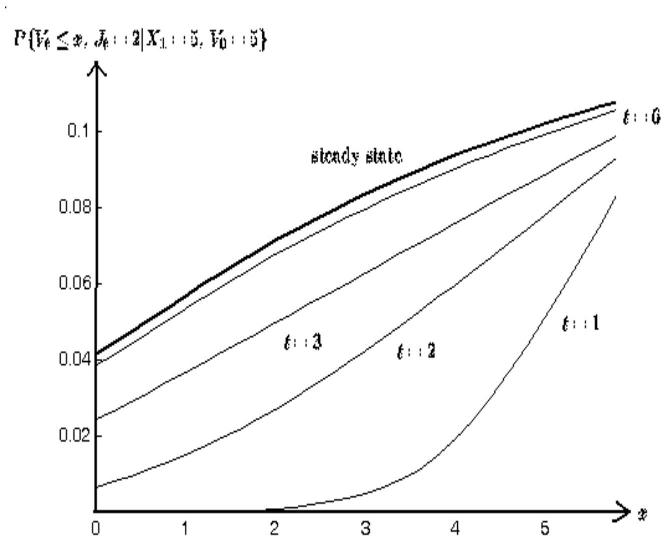

Figure 6.13: The time-dependent distribution function $F_{52}(x, t, 5)$ for different values of t, $\rho = 0.822$, the net input rates: $r_1 = -2.16, r_2 = -1.41, r_3 = -3.66, r_4 = 1.33, r_5 = 3.33$.

x	$F_{52}(x, 1, 5)$	$F_{52}(x, 2, 5)$	$F_{52}(x, 3, 5)$	$F_{52}(x, 6, 5)$	$P\{V \leq x, J = 2\}$
0	0	0.00357	0.02078	0.03825	0.04635
0.6	4.0e-010	0.00756	0.02801	0.04738	0.05692
1	9.1e-009	0.01119	0.03590	0.05390	0.06442
1.6	1.3e-005	0.01808	0.04402	0.06331	0.07510
2	0.00021	0.02363	0.04955	0.06915	0.08158
2.6	0.00157	0.03313	0.05799	0.07720	0.09029
3	0.00355	0.04007	0.06370	0.08213	0.09544
3.6	0.01013	0.05110	0.07232	0.08892	0.10226
4	0.01684	0.05877	0.07807	0.09310	0.10627
4.6	0.03705	0.07058	0.08377	0.09892	0.11157
5	0.05187	0.07854	0.08939	0.10253	0.11467
5.6	0.07629	0.09039	0.09762	0.10757	0.11876

Table 6.12: The values of $F_{52}(x, t, 5)$ and the steady-state distribution function in figure 6.13 on some values of x

Appendix A

Appendix

A.1 Cauchy's Integral Formula

Theorem A.1.1 (Cauchy's integral formula)
Let f be an analytic function on a open set Ω, an let γ be any circuit which is homomorphic to a point in Ω. Then for any point z in Ω which is not on the graph of γ we have

$$n(\gamma, z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw, \tag{A.1}$$

where $n(\gamma, z)$ is called the winding number of γ with respect to z.

We refer to Apostol [5] for the definition of $n(\gamma, z)$. If $\gamma(t)$ has domain $[a, b]$, then $n(\gamma, z)$ gives us the number of times the point $\gamma(t)$ winds around the point z as t varies over the interval $[a, b]$. For example, if γ is a positively oriented circle given by $\gamma(\theta) = z + re^{i\theta}$, where $0 \le \theta \le 2\pi$, then the winding number is 1. For the proof see also [5].

A.2 Some Limit Theorems

Theorem A.2.1 (Abel's theorem for power series)
If $\sum_{n=0}^\infty a_n r^n$ converges for $|r| < 1$ and $\lim_{n\to\infty} a_n = a$, then

$$\lim_{r\uparrow 1}(1-r)\sum_{n=0}^\infty a_n r^n = a.$$

Proof. See Feller [26].

Theorem A.2.2 (Abel's theorem for Laplace transforms)
If $\int_0^\infty a(t)dt$ is convergent then $\int_0^\infty e^{-\phi t}a(t)dt$ is uniformly convergent for $Re(\phi) \ge 0$ and

$$\lim_{\phi\to 0}\int_0^\infty e^{-\phi t}a(t)dt = \int_0^\infty a(t)dt, \quad |arg(\phi)| \le \theta < \frac{1}{2}\pi.$$

If $\int_0^\infty a(t)dt$ *is convergent for* $Re(\phi) \geq 0$ *and* $a(t)$ *has a limit for* $t \to \infty$ *then*

$$\lim_{t\to\infty} a(t) = \lim_{\phi\to 0} \phi \int_0^\infty e^{-\phi t} a(t)dt, \quad |arg(\phi)| \leq \theta < \frac{1}{2}\pi.$$

Proof. See Doetsch [25].

A.3 Some inversion formulas

The following inversion formula can be found from Widder[43], page 69.

Lemma A.3.1
Let F *be the distribution function of a non-negative random variable with Laplace-Stieltjes transform* $f^*(\phi)$*. Then for* $x > 0$,

$$\frac{F(x+) + F(x-)}{2} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\phi x}}{\phi} f^*(\phi)d\phi, \quad c > 0. \tag{A.2}$$

For rational Laplace-Stieltjes transforms, we refer to Bateman & Erdelyi [11] for the inversion formula. Let

$$g(\phi) = \int_0^\infty e^{-\phi t} f(t)dt = \frac{Q(\phi)}{P(\phi)},$$

where

$$P(\phi) = (\phi - a_1)^{m_1}(\phi - a_2)^{m_2} \cdots (\phi - a_n)^{m_n}$$

and $Q(\phi)$ is a polynomial of degree $m_1 + m_2 + \cdots + m_n$ or less, $a_i \neq a_j$ for $i \neq j$. Then the inverse is given by

$$f(t) = \sum_{k=1}^{n} \sum_{l=1}^{m_k} \frac{\Phi_{kl}(a_k) t^{m_k - 1} e^{a_k t}}{(m_k - l)!(l-1)!} \tag{A.3}$$

with

$$\Phi_{kl}(\phi) = \frac{\partial^{l-1}}{\partial \phi^{l-1}} \left(\frac{Q(\phi)}{P_k(\phi)} \right)$$

and

$$P_k(\phi) = \frac{P(\phi)}{(\phi - a_k)^{m_k}}.$$

A.4 Characteristics of the zeros of a function

Information on the location of zeros of a function is needed in order to get a unique Wiener-Hopf factorization(see subsection 2.3).

Theorem A.4.1 (Rouché's Theorem)
Let f *and* g *be analytic in* D *and let* γ *be a simply closed path which is null homologous in* D *and which satisfies*

$$|f(\zeta) - g(\zeta)| < |g(\zeta)| \quad \text{for all} \quad \zeta \in \gamma.$$

Then f *and* g *have the same number of zeros inside* γ.

Proof. See Apostol [5].

For the queueing systems in chapters 4-6, the functions we consider are actually the determinants of certain polynomial matrices. The generalization of Rouché's theorem for those determinants is given in de Smit [21].

Theorem A.4.2 (Generalization of Rouché's Theorem)

Let $\mathbf{A}(\phi) = (a_{ij}(\phi))$ and $\mathbf{B}(\phi) = (b_{ij}(\phi))$ be complex $n \times n-$matrices, where $\mathbf{B}(\phi)$ is diagonal. The elements a_{ij} and $b_{ij}, 1 \leq i \leq n, 1 \leq j \leq n$, are meromorphic functions in a simply connected region S in which T is the set of all poles of these functions. C is a rectifiable closed Jordan curve in $S - T$. $N_{\mathbf{A}}[N_{\mathbf{A+B}}]$ is the number of zeros inside C of $\det \mathbf{B}(\phi)[\det(\mathbf{A}(\phi) + \mathbf{B}(\phi))]$ and $P_{\mathbf{A}}[P_{\mathbf{A+B}}]$ is the number of poles inside C(poles and zeros of higher order are counted according to this order). If

(i) $|b_{ii}(\phi)| > \sum_{j=1}^{n} |a_{ij}(\phi)|$ *on* C *for all* $i = 1, \cdots, n$

or

(ii)$\mathbf{A}(\phi)$ *is decomposable on* C *and* $|b_{ii}(\phi)| \geq \sum_{j=1}^{n} |a_{ij}(\phi)|$ *on* C, *for all* $i = 1, \cdots, n$ *with strict inequality for at least one* i,

then on C

$$\det(\mathbf{A}(\phi) + \mathbf{B}(\phi)) \neq 0,$$

$$\det \mathbf{B}(\phi) \neq 0,$$

and

$$N_{\mathbf{A+B}} - P_{\mathbf{A+B}} = N_{\mathbf{B}} - P_{\mathbf{B}}.$$

A.5 Some results from the Theory of Matrices

The following lemma concerns the inverse of a simple constant matrix. An $N \times N$-dimensional matrix $\mathbf{A}$ is called a simple matrix if it has N eigenvalues which are all distinct.

Lemma A.5.1

Let $\mathbf{A}_0$ be a simple $N \times N$-dimensional matrix. Let $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N$ be the right eigenvectors corresponding to eigenvalues $a_1, a_2, \ldots, a_N$. If $\mathbf{x}_i^{-1}$ denotes the i-th row of the matrix

$$\mathbf{X}^{-1} = \left(\mathbf{x}^1 \mathbf{x}^2 \ldots \mathbf{x}^N\right)^{-1}$$

then the inverse of matrix $\phi\mathbf{I} - \mathbf{A}_0$, where ϕ is a complex number, is given by

$$(\phi\mathbf{I} - \mathbf{A}_0)^{-1} = \sum_{i=1}^{N} \frac{\mathbf{x}^i \mathbf{x}_i^{-1}}{(\phi - a_i)}.$$

Proof. Let $\mathbf{y}^i, i = 1, 2, \cdots, n$ be the left eigenvectors of $\mathbf{A}_0$. Let

$$\mathbf{X} = [\mathbf{x}^1 \mathbf{x}^2 \ldots \mathbf{x}^n],$$

$$\mathbf{Y} = [\mathbf{y}^1 \mathbf{y}^2 \ldots \mathbf{y}^n],$$

and

$$\mathbf{U} = \mathrm{diag}(a_1, a_2, \ldots, a_n).$$

Then

$$\mathbf{A}_0\mathbf{X} = \mathbf{X}\mathbf{U} \text{ and } \mathbf{Y}^t\mathbf{A}_0 = \mathbf{U}\mathbf{Y}^t.$$

Since $\mathbf{A}_0$ is simple, then $\mathbf{X}$ is nonsingular . Hence,

$$\mathbf{X}^{-1}\mathbf{A}_0 = \mathbf{U}\mathbf{X}^{-1}. \tag{A.4}$$

It follows that the rows of $\mathbf{X}^{-1}$ are the left eigenvectors of $\mathbf{A}_0$. Hence, $\mathbf{Y}$ can be chosen so that $\mathbf{Y}^t = \mathbf{X}^{-1}$ or

$$\mathbf{Y}^t\mathbf{X} = \begin{bmatrix} \mathbf{y}^{1t} \\ \mathbf{y}^{2t} \\ \vdots \\ \mathbf{y}^{nt} \end{bmatrix} [\mathbf{x}^1\mathbf{x}^2 \ldots \mathbf{x}^n] = \mathbf{I}$$

or

$$\mathbf{y}^{it}\mathbf{x}^j = \delta_{ij}, \quad i, j = 1, 2, \ldots, n.$$

Define for $i = 1, 2, \ldots, n$ the constituent matrices

$$\mathbf{G}_i = \mathbf{x}^i\mathbf{y}_i^t$$

where $\mathbf{y}_i^t$ denotes the i-th row of the matrix $\mathbf{Y}^t$. According to these relations, we see that the constituent matrices have the following properties;

1. $\sum_{i=1}^{n} \mathbf{G}_i = \mathbf{I}$
2. $\mathbf{G}_i\mathbf{G}_j = 0, \quad i \neq j, \quad i, j = 1, 2, \ldots, n$
3. $\mathbf{G}_i^2 = \mathbf{G}_i, \quad i = 1, 2, \ldots, n.$

Now, from (A.4) we have

$$\begin{aligned} (\phi\mathbf{I} - \mathbf{A}_0) &= (\phi\mathbf{I} - \mathbf{X}\mathbf{U}\mathbf{Y}) \\ &= \left(\phi\mathbf{I} - [\mathbf{x}^1\mathbf{x}^2 \ldots \mathbf{x}^n] \begin{bmatrix} a_1\mathbf{y}_1^t \\ a_2\mathbf{y}_2^t \\ \vdots \\ a_n\mathbf{y}_n^t \end{bmatrix}\right) \\ &= \left(a\sum_{i=1}^{n} \mathbf{G}_i - \sum_{i=1}^{n} a_i\mathbf{G}_i\right) \\ &= \sum_{i=1}^{n} (\phi - a_i)\mathbf{G}_i. \end{aligned}$$

Define

$$\mathbf{B} = \sum_{l=1}^{n} \frac{\mathbf{G}_l}{(\phi - a_l)}.$$

Then by the properties of $\mathbf{G}_i$,

$$(\phi\mathbf{I} - \mathbf{A}_0)\mathbf{B} = \sum_{i=1}^{n}\sum_{l=1}^{n}\frac{(\phi - a_i)}{(\phi - a_l)}\mathbf{G}_i\mathbf{G}_l = \sum_{i=1}^{n}\mathbf{G}_i^2 = \mathbf{I}.$$

We can conclude that

$$\mathbf{B} = \sum_{l=1}^{n}\frac{\mathbf{G}_l}{(\phi - a_l)} = (\phi\mathbf{I} - \mathbf{A}_0)^{-1}.$$

■

In chapter 6 we need to invert some *matrix polynomials* or sometimes known as ϕ-matrices of degree $l, l \geq 2$ of the form

$$\mathbf{A}(\phi) = \sum_{i=0}^{l}\phi^i\mathbf{A}_i, \tag{A.5}$$

where $\mathbf{A}_0, \mathbf{A}_1, \cdots, \mathbf{A}_l$ are $N \times N$-matrices of complex numbers independent of ϕ. If $\mathbf{A}_l = \mathbf{I}_N$, the $N \times N$-identity matrix, then the matrix polynomial is said to be *monic*, and (A.5) becomes

$$\mathbf{A}(\phi) = \phi^l\mathbf{I}_N + \sum_{i=0}^{l-1}\phi^i\mathbf{A}_i. \tag{A.6}$$

The complex number ϕ_0 is called an *eigenvalue* of the monic matrix polynomial $\mathbf{A}(\phi)$ of form (A.6) if there are N-dimensional vectors $\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_k$, for some $k \geq 0$ with $\mathbf{x}_0 \neq 0$, so that for $i = 0, 1, \cdots, k$,

$$\sum_{p=0}^{i}\frac{1}{p!}\mathbf{A}^{(p)}(\phi_0)\mathbf{x}_{i-p} = 0, \tag{A.7}$$

where $\mathbf{A}^{(p)}(\phi)$ denotes the pth derivative of $\mathbf{A}(\phi)$ with respect to ϕ. The sequence of vectors $\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_k$ is called a *Jordan chain* of *length* $k+1$ of $\mathbf{A}(\phi)$ corresponding to the eigenvalue ϕ_0. The vector $\mathbf{x}_0$ is called an *eigenvector* and the subsequent vectors $\mathbf{x}_1, \cdots, \mathbf{x}_k$ are called *generalized eigenvectors*. The monic polynomial of form (A.6) has exactly Nl eigenvalues when counted with multiplicities. The set

$$\sigma(A) = \{\phi_0 | \phi_0 \text{ is the eigenvalue of } \mathbf{A}(\phi)\}$$

is called the *spectrum* of $A(\phi)$. It is clear that $\sigma(A)$ contains at most lN elements.

For $\phi \notin \sigma(\mathbf{A})$, the inverse of the monic matrix polynomial $\mathbf{A}(\phi)$ in (A.6) can be expressed in terms of the so called *standard triple* of $\mathbf{A}(\phi)$. The theorem in the following and some explanations after that will explain the standard triple of $\mathbf{A}(\phi)$ precisely.

Theorem A.5.1
For every $\phi \notin \sigma(\mathbf{A})$,

$$\mathbf{A}^{-1}(\phi) = \mathbf{P}_1(\phi\mathbf{I}_{Nl} - C_A)^{-1}\mathbf{R}_1,$$

where

$$\mathbf{P}_1 = (\mathbf{I}_N\,\mathbf{0}\cdots\mathbf{0}), \quad R_1 = \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{I}_N \end{pmatrix},$$

and

$$\mathbf{C}_A = \begin{pmatrix} \mathbf{0} & \mathbf{I}_N & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_N & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \mathbf{I}_N \\ -\mathbf{A}_0 & -\mathbf{A}_1 & -\mathbf{A}_2 & \cdots & -\mathbf{A}_{l-1} \end{pmatrix},$$

which is called the (first) companion matrix of $\mathbf{A}(\phi)$.

Proof. See page 58 of Gohberg[28].

Any three matrices $(\mathbf{U}, \mathbf{T}, \mathbf{V})$ are said to be *admissible* for $\mathbf{A}(\phi)$ if they are of size $N \times lN, lN \times lN$, and $lN \times N$, respectively. Any admissible triple $(\mathbf{U}, \mathbf{T}, \mathbf{V})$ which is *similar* to the triple $(\mathbf{P}_1, \mathbf{C}_A, \mathbf{R}_1)$ in Theorem A.5.1 above is said to be a *standard triple* for $\mathbf{A}(\phi)$. The similarity of those triples means that there is a non singular matrix $\mathbf{S}$ so that

$$\mathbf{U} = \mathbf{PS}, \quad \mathbf{T} = \mathbf{S}^{-1}\mathbf{C}_A\mathbf{S}, \quad \mathbf{V} = \mathbf{S}^{-1}\mathbf{R}_1.$$

The following theorem is a generalization of Theorem A.5.1, in the sense that the inverse of the monic matrix polynomial $\mathbf{A}(\phi)$ can be expressed by any standard triple of $\mathbf{A}(\phi)$.

Theorem A.5.2
If $(\mathbf{U}, \mathbf{T}, \mathbf{V})$ *is a standard triple for* $\mathbf{A}(\phi)$ *and* $\phi \notin \sigma(\mathbf{A})$, *then*

$$\mathbf{A}^{-1}(\phi) = \mathbf{U}(\phi\mathbf{I}_{Nl} - T)^{-1}\mathbf{V}. \tag{A.8}$$

Proof. See [28].

In chapter 6 we use the representation of $\mathbf{L}(1, \phi, \eta)^{-1}$ in form of (A.8), where $\mathbf{U} = \mathbf{E}(1, \eta)$ and $\mathbf{T} = \text{diag}(\gamma_1(1, \eta), \cdots, \gamma_{Nm}(1, \eta))$. The matrix $\begin{pmatrix} \mathbf{U} \\ \mathbf{UT} \\ \vdots \\ \mathbf{UT}^{l-1} \end{pmatrix}$ is nonsingular, and the third matrix $\mathbf{V}$ can be defined by

$$\mathbf{V} = \begin{pmatrix} \mathbf{U} \\ \mathbf{UT} \\ \vdots \\ \mathbf{UT}^{l-1} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{I}_N \end{pmatrix}.$$

The standard triple $(\mathbf{U}, \mathbf{T}, \mathbf{V})$ of the monic polynomial matrix (A.5) has the property (see page 52 of [28] for the proof) that

$$\mathbf{U}\mathbf{T}^i\mathbf{V} = \begin{cases} \mathbf{0} & , \text{for } i = 0, \cdots, l-2, \\ \mathbf{I}_N & , \text{for } i = l-1. \end{cases}$$

A.6 The proof of Lemma 5.3.3

If $\mathbf{B}$ and $\mathbf{A}$ are $n \times n$-dimensional nonsingular matrices, $\mathbf{X}$ is an $n \times r$-dimensional matrix, $\mathbf{R}$ is an $r \times r$-dimensional nonsingular matrix, and $\mathbf{Y}$ is an $r \times n$-dimensional matrix such that

$$\mathbf{B} = \mathbf{A} + \mathbf{X}\mathbf{R}\mathbf{Y},$$

then it is easy to prove(see [33]) that

$$\mathbf{B}^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{X}(\mathbf{R}^{-1} + \mathbf{Y}\mathbf{A}^{-1}\mathbf{X})^{-1}\mathbf{Y}\mathbf{A}^{-1}. \tag{A.9}$$

If we apply this result to the matrix $K(1, \phi, \eta)$ defined in (5.16), we get

$$\mathbf{K}(1, \phi, \eta)^{-1} = \mathbf{I} - \mathbf{D}(1, \eta)\mathbf{X}(\phi, \eta)^{-1}\mathbf{C}(1, \eta) \tag{A.10}$$

where

$$\begin{aligned} \mathbf{X}(\phi, \eta) =& \text{diag}(\phi - \mu_1(1, \eta), \ldots, \phi - \mu_{\bar{K}}(1, \eta)) + \mathbf{C}(1, \eta)\mathbf{D}(1, \eta) \\ =& \phi\mathbf{I} + (\mathbf{I}_{\bar{K}N}\mathbf{E}(1, \eta)\mathbf{I}_{N\bar{K}})^{-1} I_{\bar{K}N}\alpha(\mathbf{I} + \eta\mathbf{q}^{-1})\mathbf{E}(1, \eta)\mathbf{I}_{N\bar{K}}. \end{aligned}$$

The inverse of $\mathbf{X}(\phi, \eta)$, by using the same relation, is

$$\frac{1}{\phi}\mathbf{I} - \frac{1}{\phi}(\mathbf{I}_{\bar{K}N}\mathbf{E}(1, \eta)\mathbf{I}_{N\bar{K}})^{-1}\mathbf{I}_{\bar{K}N}\boldsymbol{\alpha}(\eta)\mathbf{M}(\phi, \eta)^{-1}\mathbf{E}(1, \eta)\mathbf{I}_{N\bar{K}}.$$

If we substitute this into (A.10) we get

$$\begin{aligned} \mathbf{K}(1, \phi, \eta)^{-1} =& \mathbf{I} - \frac{1}{\phi}\mathbf{D}(1, \eta)\mathbf{C}(1, \eta)\left[\mathbf{I} - \boldsymbol{\alpha}^{-1}\boldsymbol{\alpha}(\eta)\mathbf{M}(\phi, \eta)^{-1}\mathbf{E}(1, \eta)\mathbf{I}_{N\bar{K}}\mathbf{C}(1, \eta)\right] \\ =& \mathbf{I} - \mathbf{D}(1, \eta)\mathbf{C}(1, \eta)\mathbf{M}(\phi, \eta)^{-1}, \end{aligned}$$

and we get (5.24). Since the last $N - \bar{K}$ columns of $\mathbf{C}(1, \eta)$ are zero, it is clear that $\mathbf{K}^{-1}(1, \phi, \eta)$ is analytic in the left half-plane $Re(\phi) < 0$.

Moreover, from equation (5.8) we get

$$\mathbf{H}^{-1}(1, \phi, \eta) = \mathbf{M}(\phi, \eta)\boldsymbol{\alpha}^{-1}\mathbf{L}^{-1}(1, \phi, \eta)\boldsymbol{\alpha}. \tag{A.11}$$

Since for $i \in \mathcal{N}$ $\bar{\mu}_i$ and $\bar{E}_i$ satisfy the equation (5.12), then by Condition 5.3.1 and Lemma A.5.1 we have

$$\mathbf{L}^{-1}(1, \phi, \eta) = \sum_{i=1}^{N} \frac{\bar{\mathbf{E}}^i\bar{\mathbf{E}}_i^{-1}}{(\phi - \bar{\mu}_i)}, \tag{A.12}$$

and if we substitute this into (A.11) we get (5.25).

■

A.7 The proof of Lemma 5.4.1

Let

$$\tilde{\boldsymbol{E}}(z,\eta) = (\mathbf{I}_{\bar{K}N}\mathbf{E}(z,\eta)\mathbf{I}_{N\bar{K}})^{-1}\mathbf{I}_{\bar{K}N}.$$

Then by the various definitions,

$$\begin{aligned}
&\mathbf{M}(\phi,\eta) - \mathbf{D}(z,\eta)\mathbf{C}(z,\eta)\\
=&\phi\mathbf{I} + \boldsymbol{\alpha}(\eta) - (\mathbf{I} + \eta\boldsymbol{q}^{-1})\mathbf{E}(z,\eta)\mathbf{I}_{N\bar{K}}\tilde{\boldsymbol{E}}(z,\eta)\boldsymbol{\alpha}\\
&- \boldsymbol{\alpha}^{-1}\mathbf{E}(z,\eta)\mathbf{I}_{N\bar{K}}\mathrm{diag}(\mu_1(z,\eta),\cdots,\mu_{\bar{K}}(z,\eta))\tilde{\boldsymbol{E}}(z,\eta)\boldsymbol{\alpha}\\
=&\phi\mathbf{I} + \boldsymbol{\alpha}(\eta) - (\mathbf{I} + \eta\boldsymbol{q}^{-1})\begin{pmatrix}\mathbf{I}_{\bar{K}} & 0\\ \mathbf{A} & 0\end{pmatrix}\boldsymbol{\alpha}\\
&- \boldsymbol{\alpha}^{-1}\sum_{k=1}^{\bar{K}}\mu_k(z,\eta)\mathbf{E}^k(z,\eta)\tilde{\boldsymbol{E}}_k(z,\eta)\boldsymbol{\alpha},
\end{aligned}$$

where $\mathbf{I}_{\bar{K}}$ is $\bar{K}\times\bar{K}$-dimension identity matrix and $\mathbf{A}$ is $(N-\bar{K})\times\bar{K}$-dimension matrix with elements

$$A_{ij} = (\mathbf{E}(z,\eta)\mathbf{I}_{N\bar{K}})_{\bar{K}+i}\tilde{\boldsymbol{E}}^j(z,\eta).$$

It follows that

$$\begin{aligned}
&\mathbf{C}_i(z,\eta)\left[\mathbf{M}(\phi,\eta) - \mathbf{D}(z,\eta)\mathbf{C}(z,\eta)\right]\\
=&\tilde{\boldsymbol{E}}_i(z,\eta)\boldsymbol{\alpha}\left(\phi\mathbf{I} + \boldsymbol{\alpha}(\eta)\right) - \tilde{\boldsymbol{E}}_i(z,\eta)\boldsymbol{\alpha}(\eta)\begin{pmatrix}\mathbf{I}_{\bar{K}} & 0\\ \mathbf{A} & 0\end{pmatrix}\boldsymbol{\alpha}\\
&- \tilde{\boldsymbol{E}}_i(z,\eta)\sum_{k=1}^{\bar{K}}\mu_k(z,\eta)\mathbf{E}^k(z,\eta)\tilde{\boldsymbol{E}}_k(z,\eta)\boldsymbol{\alpha} \qquad (A.13)\\
=&\phi\tilde{\boldsymbol{E}}_i(z,\eta)\boldsymbol{\alpha} + \tilde{\boldsymbol{E}}_i(z,\eta)\boldsymbol{\alpha}\boldsymbol{\alpha}(\eta) - \tilde{\boldsymbol{E}}_i(z,\eta)\boldsymbol{\alpha}(\eta)\begin{pmatrix}\mathbf{I}_{\bar{K}} & 0\\ \mathbf{A} & 0\end{pmatrix}\boldsymbol{\alpha}\\
&- \mu_i(z,\eta)\tilde{\boldsymbol{E}}_i(z,\eta)\boldsymbol{\alpha}.
\end{aligned}$$

The second and the third term of the last equation in (A.13) will cancel since the last $N-\bar{K}$ elements of $\tilde{\boldsymbol{E}}_i(z,\eta)$ are zero. We then can conclude that

$$\mathbf{C}_i(z,\eta)\left[\mathbf{M}(\phi,\eta) - \mathbf{D}(z,\eta)\mathbf{C}(z,\eta)\right] = (\phi - \mu_i(z,\eta))\mathbf{C}_i(z,\eta).$$

■

A.8 The proof of Lemma 5.5.1

From Lemma 5.3.3 now we have

$$\mathbf{K}^{-1}(1,\phi,\eta)\mathbf{H}^{-1}(1,\phi,\eta) = \left(\mathbf{M}(\phi,\eta) - \mathbf{D}(1,\eta)\mathbf{C}(1,\eta)\right)\boldsymbol{\alpha}^{-1}\sum_{i=1}^{N}\frac{\bar{\mathbf{E}}^i\bar{\mathbf{E}}_i^{-1}}{(\phi-\bar{\mu}_i)}\boldsymbol{\alpha}, \qquad (A.14)$$

and based on this result we will derive the expression for $\mathbf{Z}^*(\phi,\eta,v)$.

By definition, we have

$$\begin{aligned}
&\mathbf{K}^+(1,\phi,\eta,v)+\mathbf{K}^-(1,0,\eta,v)\\
=&\mathbf{Z}^0(\phi,\eta,v)\mathbf{K}(1,\phi,\eta,v)-\big(\mathbf{K}^-(1,\phi,\eta,v)-\mathbf{K}^-(1,0,\eta,v)\big)\\
=&\mathbf{Z}^0(\phi,\eta,v)\mathbf{K}(1,\phi,\eta,v)-\phi\sum_{i=1}^{\bar{K}}\frac{\mathbf{Z}^0(\bar{\mu}_i,\eta,v)\mathbf{D}^i(1,\eta)\mathbf{C}_i(1,\eta)}{\bar{\mu}_i(\phi-\bar{\mu}_i)},
\end{aligned}$$

so that

$$\begin{aligned}
&\big(\mathbf{K}^+(1,\phi,\eta,v)+\mathbf{K}^-(1,0,\eta,v)\big)\,\mathbf{K}(1,\phi,\eta,v)^{-1}H(1,\phi,\eta)^{-1}\\
=&\mathbf{Z}^0(\phi,\eta,v)\mathbf{H}(1,\phi,\eta)^{-1}\\
&-\phi\sum_{i=1}^{\bar{K}}\frac{\mathbf{Z}^0(\bar{\mu}_i,\eta,v)\mathbf{D}^i(1,\eta)\mathbf{C}_i(1,\eta)}{\bar{\mu}_i(\phi-\bar{\mu}_i)}\mathbf{K}(1,\phi,\eta,v)^{-1}\mathbf{H}(1,\phi,\eta)^{-1}.
\end{aligned}\tag{A.15}$$

By combining (A.15) with Lemma 5.4.1 and (A.14), we obtain

$$\begin{aligned}
&\big(\mathbf{K}^+(1,\phi,\eta,v)+\mathbf{K}^-(1,0,\eta,v)\big)\,\mathbf{K}^{-1}(1,\phi,\eta,v)\mathbf{H}^{-1}(1,\phi,\eta)\\
=&\mathbf{Z}^0(\phi,\eta,v)\mathbf{M}(\phi,\eta)\boldsymbol{\alpha}^{-1}\sum_{l=1}^{N}\frac{\bar{\mathbf{E}}^l\bar{\mathbf{E}}_l^{-1}}{(\phi-\bar{\mu}_l)}\boldsymbol{\alpha}\\
&-\phi\sum_{l_1=1}^{\bar{K}}\frac{\mathbf{Z}^0(\bar{\mu}_{l_1},\eta,v)}{\bar{\mu}_{l_1}}\mathbf{M}(\bar{\mu}_{l_1},\eta)\boldsymbol{\alpha}^{-1}\bar{\mathbf{E}}^{l_1}\mathbf{C}_{l_1}(1,\eta)\boldsymbol{\alpha}^{-1}\sum_{l_2=1}^{N}\frac{\bar{\mathbf{E}}^{l_2}\bar{\mathbf{E}}_{l_2}^{-1}}{(\phi-\bar{\mu}_{l_2})}\boldsymbol{\alpha}\\
=&\mathbf{Z}^0(\phi,\eta,v)\mathbf{M}(\phi,\eta)\boldsymbol{\alpha}^{-1}\sum_{l=1}^{N}\frac{\bar{\mathbf{E}}^l\bar{\mathbf{E}}_l^{-1}}{(\phi-\bar{\mu}_l)}\boldsymbol{\alpha}\\
&-\sum_{l=1}^{\bar{K}}\frac{\mathbf{Z}^0(\bar{\mu}_l,\eta,v)}{\bar{\mu}_l}\mathbf{M}(\bar{\mu}_l,\eta)\boldsymbol{\alpha}^{-1}\bar{\mathbf{E}}^l\mathbf{C}_l(1,\eta)\\
&-\sum_{l=1}^{\bar{K}}\frac{\mathbf{Z}^0(\bar{\mu}_l,\eta,v)}{\bar{\mu}_l}\mathbf{M}(\bar{\mu}_l,\eta)\boldsymbol{\alpha}^{-1}\bar{\mathbf{E}}^l\mathbf{C}_l(1,\eta)\boldsymbol{\alpha}^{-1}\sum_{l_1=1}^{N}\frac{\bar{\mathbf{E}}^{l_1}\bar{\mathbf{E}}_{l_1}^{-1}}{(\phi-\bar{\mu}_{l_1})}\boldsymbol{\alpha}.
\end{aligned}\tag{A.16}$$

If we substitute (A.16) into (5.42) then we get (5.55).

■

A.9 The proof of Lemma 6.3.4

By the definitions of $\mathbf{C}(1,\eta)$ and $\mathbf{D}(1,\eta)$ we have for $Re(\eta) \geq 0$,

$$\begin{aligned}&\mathbf{C}(1,\eta)\mathbf{D}(1,\eta)\\ =&\mathbf{S}(1,\eta)^{-1}\mathbf{C}_0\mathbf{D}(1,\eta)\\ =&\mathbf{S}(1,\eta)^{-1}\mathbf{C}_0\left(\mathbf{M}(\gamma_1(1,\eta),\eta)\mathbf{E}^1(1,\eta)\cdots\mathbf{M}(\gamma_{\bar{K}m}(1,\eta),\eta)\mathbf{E}^{\bar{K}m}(1,\eta)\right),\end{aligned}$$

where the (i,j)th element of

$$\mathbf{M}(\gamma_1(1,\eta),\eta)\mathbf{E}^1(1,\eta)\cdots\mathbf{M}(\gamma_{\bar{K}m}(1,\eta),\eta)\mathbf{E}^{\bar{K}m}(1,\eta)$$

is equal

$$\prod_{k=1}^{m}(\gamma_j(1,\eta)+\alpha_{ik}(\eta))E_{ij}(1,\eta),$$

so that

$$\begin{aligned}&\mathbf{C}_0\left(\mathbf{M}(\gamma_1(1,\eta),\eta)\mathbf{E}^1(1,\eta)\cdots\mathbf{M}(\gamma_{\bar{K}m}(1,\eta),\eta)\mathbf{E}^{\bar{K}m}(1,\eta)\right)\\ =&\,\mathbf{S}(1,\eta)\widetilde{\boldsymbol{\gamma}}(1,\eta)+\widetilde{\boldsymbol{\alpha}(\eta)}\mathbf{S}(1,\eta),\end{aligned}$$

where

$$\widetilde{\boldsymbol{\gamma}}(1,\eta)=\operatorname{diag}(\gamma_1(1,\eta),\cdots,\gamma_{\bar{K}m}(1,\eta))$$

and

$$\widetilde{\boldsymbol{\alpha}}(\eta)=\operatorname{diag}(\alpha_{11}(\eta),\alpha_{21}(\eta),\cdots,\alpha_{\bar{K}1}(\eta),\alpha_{12}(\eta),\cdots,\alpha_{\bar{K}2}(\eta),\cdots,\alpha_{\bar{K}m}(\eta)).$$

It follows that

$$\begin{aligned}\mathbf{X}(\phi,\eta)&=\phi\mathbf{I}-\widetilde{\boldsymbol{\gamma}}(\eta)+\mathbf{C}(1,\eta)\mathbf{D}(1,\eta)\\ &=\phi\mathbf{I}-\mathbf{S}(1,\eta)^{-1}\mathbf{S}(1,\eta)\widetilde{\boldsymbol{\gamma}}(1,\eta)+\mathbf{S}(1,\eta)^{-1}\mathbf{S}(1,\eta)\widetilde{\boldsymbol{\gamma}}(1,\eta)\\ &\quad+\mathbf{S}(1,\eta)^{-1}\widetilde{\boldsymbol{\alpha}(\eta)}\mathbf{S}(1,\eta)\\ &=\phi\mathbf{I}+\mathbf{S}(1,\eta)^{-1}\widetilde{\boldsymbol{\alpha}(\eta)}\mathbf{S}(1,\eta),\end{aligned}$$

and

$$\mathbf{X}(\phi,\eta)^{-1}=\mathbf{S}(1,\eta)^{-1}(\phi\mathbf{I}+\widetilde{\boldsymbol{\alpha}}(\eta))^{-1}\mathbf{S}(1,\eta), \tag{A.17}$$

and it proves the lemma.

A.10 Analyticity of $Z(1,\phi,\eta,v)$ at $\phi = \gamma_i(\eta)$ for $i = 1,\cdots,\bar{K}m$

We see that for $l = 1,2,\cdots,\bar{K}m$,

$$\begin{aligned}
&\lim_{\phi\to\gamma_l(1,\eta)}(\phi-\gamma_l(1,\eta))\mathbf{Z}(1,\phi,\eta,v)\\
=&\mathbf{Z}^0(\gamma_l(1,\eta),\eta,v)\mathbf{M}(\gamma_l(1,\eta),\eta)\mathbf{E}^l(\eta)\mathbf{Y}_l(\eta)\\
&-\gamma_l(1,\eta)\sum_{i=1}^{\bar{K}m}\frac{\mathbf{Z}^0(\gamma_i(1,\eta),\eta,v)\mathbf{D}^i(1,\eta)}{\gamma_i(1,\eta)}\mathbf{1}_i\mathbf{X}^{-1}(\gamma_l(1,\eta),\eta)\mathbf{C}(1,\eta)\mathbf{M}(\gamma_l(1,\eta),\eta)\\
=&\mathbf{Z}^0(\gamma_l(1,\eta),\eta,v)\mathbf{M}(\gamma_l(1,\eta),\eta)\mathbf{E}^l(\eta)\mathbf{Y}_l(\eta)\\
&-\gamma_l(1,\eta)\sum_{i=1}^{\bar{K}m}\frac{\mathbf{Z}^0(\gamma_i(1,\eta),\eta,v)\mathbf{D}^i(1,\eta)}{\gamma_i(1,\eta)}\mathbf{1}_i\mathbf{X}^{-1}(\gamma_l(1,\eta),\eta)\mathbf{C}(1,\eta)\mathbf{D}^l(1,\eta)\mathbf{Y}_l(\eta)\\
&.\mathbf{E}^l(\eta)\mathbf{Y}_l(\eta),
\end{aligned}$$

using $\mathbf{D}^l(1,\eta) = \mathbf{M}(\gamma_l(1,\eta),\eta)\mathbf{E}^l(\eta)$. Eliminating $\mathbf{C}(1,\eta)\mathbf{D}^l(1,\eta)$ using (6.31) we obtain for $Re(\eta) > 0$,

$$\begin{aligned}
&\lim_{\phi\to\gamma_l(1,\eta)}(\phi-\gamma_l(1,\eta))\mathbf{Z}(1,\phi,\eta,v)\\
=&\mathbf{Z}^0(\gamma_l(1,\eta),\eta,v)\mathbf{M}(\gamma_l(1,\eta),\eta)\mathbf{E}^l(\eta)\mathbf{Y}_l(1,\eta)\\
&-\gamma_l(1,\eta)\sum_{i=1}^{\bar{K}m}\frac{\mathbf{Z}^0(\gamma_i(1,\eta),\eta,v)\mathbf{D}^i(1,\eta)}{\gamma_i(1,\eta)}\mathbf{1}_i\mathbf{X}^{-1}(\gamma_l(1,\eta),\eta)\\
&(\mathbf{C}(1,\eta)\mathbf{D}^l(1,\eta)+(\gamma_l(1,\eta)-\gamma_l(1,\eta))\mathbf{1}_l^T)\mathbf{Y}_l(\eta)\\
=&\mathbf{Z}^0(\gamma_l(1,\eta),\eta,v)\mathbf{M}(\gamma_l(1,\eta),\eta)\mathbf{E}^l(\eta)\mathbf{Y}_l(\eta)\\
&-\gamma_l(1,\eta)\sum_{i=1}^{\bar{K}m}\frac{\mathbf{Z}^0(\gamma_i(1,\eta),\eta,v)\mathbf{D}^i(1,\eta)}{\gamma_i(1,\eta)}\mathbf{1}_i\mathbf{1}_l^T\mathbf{Y}_l(\eta)\\
=&\mathbf{0}
\end{aligned}\tag{A.18}$$

or $\mathbf{Z}(1,\phi,\eta,v)$ has no poles at $\gamma_l(1,\eta)$ for $l = 1,2,\ldots,\bar{K}m$.

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Index

Summary

In this thesis we study the time-dependent behavior of queueing systems. The study is focused on the queueing systems:

1. the $GI/G/1$ system,

2. the $GI/H_m/s$ system,

3. *the Markovian Fluid Flow Model*, the fluid flow model that is modulated by a Markov process,

4. *the Semi-Markovian Fluid Flow Model*, a generalization of the Markovian Fluid Flow Model.

In general, the time-dependent behavior of queueing systems is much influenced by the initial server(s)'s work load. This leads us to consider the queueing systems with non-zero initial server(s)'s work load. In the $GI/G/1$ system and the $GI/H_m/s$ system this means that in the beginning there exist a number of (special) customers to serve. In the last two systems, initially the buffer has non-zero content.

The technique that is used to analyze the behavior of the queueing systems studied in this thesis is based on the Wiener-Hopf factorization. A brief discussion on the Wiener-Hopf factorization is given in chapter 2, where we also give the conditions on the existence of uniqueness of the factorization. In this chapter we also give some preliminaries that we need for the analysis in the rest chapters.

The first major step in the analysis is the derivation of the (system of) transformed Wiener-Hopf equation(s). Wiener-Hopf factorization is then applied to its symbol. Since the queueing systems we consider have a non-zero initial working load, the Wiener-Hopf factorization should be followed by a decomposition on a certain (matrix) function. The Wiener-Hopf factorization and the decomposition yields a (formal) solution of the (system of) equation(s).

If the stability condition is fulfilled, then the steady-state distributions of interest can be determined by applying Abel's limit theorem to the solution of the (system of) equation(s).

In chapter 3 we study the system $GI/G/1$ with non-zero initial number of customers. We get the explicit factorizations for two special systems, the systems $GI/K_n/1$ and $K_m/G/1$. These results give explicit expressions for the Lapace-Stieltjes transform of actual waiting times and virtual waiting times. Then, by applying a contour integration, we get the expectation of number of customers at arrival epochs and in continuous times as

well. At the end of this chapter we give numerical results to illustrate the behavior of the system as it tends to the steady-state.

In chapter 4 we study the system $GI/H_m/s$ with non-zero initial number of customers. As in chapter 3, the Wiener-Hopf factorization gives explicit expressions for the Lapace-Stieltjes transform of actual waiting times and virtual waiting times. Then, the distributions of the queue length and the number of customers in the system are derived, both at arrival epochs and in continuous time. At the end of this chapter we again give numerical results.

In chapter 5 we study the Markovian Fluid Flow Model, the fluid flow model in which the rate of data that flow into the buffer depends on the state of a Markov process. The Wiener-Hopf factorization gives us an explicit expression for the Laplace-Stieltjes transform of buffer content at transition epochs of the underlying Markov process. From this we can derive the distribution of the buffer content in continuous times. We conclude this chapter with some numerical results.

In chapter 6 we study a generalization of the model of chapter 5. The times between transitions of the underlying Markov process are not assumed to be exponentially distributed anymore but are assumed to be either hyper-exponentially distributed or hypo-exponentially distributed. With this assumption, the symbol of Wiener-Hopf-type equations is still a rational matrix in ϕ, and each element of this matrix has only simple poles. This matrix can be factorized by the Wiener-Hopf factorization technique as we apply in chapter 5. We have obtained the distribution of the buffer content and the corresponding numerical results.

Ringkasan

Thesis ini membahas perilaku pada waktu tertentu(*time-dependent behavior*) dan perilaku pada keadaan *steady state* dari beberapa sistem antrian. Pembahasan perilaku-perilaku tersebut difokuskan pada sistem-sistem antrian:

1. Sistem antrian $GI/G/1$,
2. Sistem antrian $GI/H_m/s$,
3. *Markovian Fluid Flow Models*, yaitu model-model *fluid flow* yang berlandaskan pada suatu proses Markov,
4. *Semi-Markovian Fluid Flow Models*, yang merupakan perumuman dari *Markovian Fluid Flow Models*.

Secara umum, perilaku suatu sistem antrian pada suatu selang waktu yang terbatas akan sangat dipengaruhi oleh keadaan awal sistem antrian tersebut: apakah pada keadaaan awal (para) pelayan yang ada sedang sibuk melayani (para) pelanggan atau tidak. Landasan pikiran ini mendorong kita untuk memandang sistem-sistem antrian di atas dengan beban kerja dari (para) pelayan pada awal pengamatan yang tidak nol. Ini berarti dalam dua sistem antrian pertama, sistem $GI/G/1$ dan $GI/H_m/s$, pada awal pengamatan terdapat sejumlah pelanggan dalam antrian. Sedangkan pada dua sistem terakhir, hal ini berarti pada awal pengamatan kita mempunyai *buffer* yang berisi.

Teknik yang dipakai untuk menentukan perilaku antrian - antrian yang dibahas dalam thesis ini adalah suatu teknik berdasarkan faktorisasi Wiener-Hopf. Penjelasan tentang faktorisasi Wiener-Hopf kami berikan di bab 2, di mana pada bab tersebut kami juga membahas kondisi-kondisi agar eksistensi dari ketunggalan faktorisasi Wiener-Hopf ini dijamin. Di bab yang sama kami juga menampilkan penjelasan singkat tentang beberapa teori dasar yang dipakai dalam menganalisa sistem-sistem antrian yang dibahas di thesis ini.

Jika kondisi kestabilan dipenuhi, maka distribusi-distribusi peluang yang menjadi perhatian pada keadaan *steady state* dapat diturunkan dengan cara menerapkan teorema limit Abel pada solusi dari sistem persamaan Wiener-Hopf. Hasilnya kemudian kita inversi secara analitik unutk mendapatkan distribusi yang kita inginkan.

Untuk sistem-sistem antrian di thesis ini, transformasi Laplace-Stieltjes ganda yang terturunkan merupakan fungsi rasional terhadap salah satu peubahnya. Hal ini memungkinkan kita melakukan inversi secara analitik untuk mendapatkan transformasi Laplace-Stieltjes tunggal. Pada transformasi tunggal inilah, yang tidak lagi berupa fungsi rasional,

kita dapat melakukan inversi secara numerik untuk melihat kelakuan sistem antrian pada saat - saat tertentu.

Di bab 3 dibahas sistem antrian $GI/G/1$ dengan jumlah pelanggan pada saat awal tidak sama dengan nol. Kita dapatkan faktor-factor eksplisit untuk dua kasus khusus sistem ini, yaitu sistem-sistem $GI/K_n/1$ dan $K_m/G/1$. Hasil ini memberikan ekspresi eksplisit transformasi Laplace-Stieltjes dari distribusi waktu tunggu sebenarnya juga waktu tunggu virtual. Sesudah itu, dengan teknik integral garis kita dapatkan ekspektasi dari jumlah pelanggan pada sistem pada titik-titik kedatangan pelanggan dan untuk waktu-waktu yang kontinu. Di akhir bab ditampilkan hasil perhitungan numerik untuk memberikan ilustrasi tentang kelakuan sistem mulai awal pengamatan sampai dicapai keadaan *steady state*.

Di bab 4 dibahas sistem antrian $GI/H_m/s$ dengan jumlah pelanggan pada awal pengamatan tidak sama dengan nol. Dengan menerapkan teknik faktorisasi Wiener-Hopf yang dilengkapi kita dapatkan ekspresi eksplisit transformasi Laplace-Stieltjes dari distribusi waktu tunggu sebenarnya juga waktu tunggu virtual. Sesudah itu, diturunkan distribusi dari panjang antrian juga banyaknya pelanggan pada sistem, keduanya dilihat pada titik-titik kedatangan pelanggan juga untuk waktu yang kontinu. Di akhir bab ditampilkan hasil perhitungan numerik untuk memberikan ilustrasi tentang kelakuan sistem mulai awal pengamatan sampai dicapai keadaan *steady state*.

Bab 5 berisi pembahasan tentang *Markovian Fluid Flow Model*, yaitu model fluid flow dimana aliran data yang masuk ke dalam *buffer* berdasarkan pada keadaan dari suatu proses Markov. Penerapan teknik faktorisasi Wiener-Hopf yang dilengkapi menghasilkan ekspresi eksplisit transformasi Laplace-Stieltjes dari isi *buffer* pada titik-titik transisi proses Markov yang mendasari proses aliran data. Dari hasil ini kita juga bisa mendapatkan distribusi untuk isi *buffer* pada waktu yang kontinu. Hasil perhitungan numerik diberikan di akhir bab untuk memberi ilustrasi tentang distribusi peluang dari isi *buffer* mulai saat awal sistem berjalan sampai keadaan steady state dicapai.

Bab 6 mempelajari perumuman dari model di bab 5. Di bab ini, waktu antar transisi dari proses Markov yang mendasari aliran data yang masuk ke *buffer* tidak lagi diasumsikan berdistribusi eksponensial, tetapi dibuat/diasumsikan memiliki distribusi *hyper-exponential* atau *hypo-exponential.* Dengan asumsi ini, matriks dari simbol Wiener-Hopf masih merupakan matriks rasional dengan elemen-elemen yang mempunyai *pole* yang berorde satu. Karena itu, teknik faktorisasi dan dekomposisi seperti di bab 5 masih bisa digunakan untuk menyelesaikan sistem persamaan Wiener-Hopf dari model di bab ini. Distribusi dari isi *buffer* didapatkan dengan teknik ini, dan hasil numerik untuk distribusi ini diberikan di akhir bab.

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About the author

Rieske Hadianti was born in Bandung, Indonesia, on 13 February 1969. She finished her study in mathematics at Institut Teknologi Bandung, in Bandung. In 1992 she started her graduate study in mathematics at Institut Teknologi Bandung, and she finished it in 1994. Her interest in mathematics has sent her in 1996 to University of Twente to do doctoral research in Queuing Theory under supervision of Prof. J.H.A. de Smit. Due to serious illness she got, she went back to Bandung in 2000 and from then she has been joining Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung as an academic staff.

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